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Abstract
In our study we consider nonlinear, power-law field-dependent electrical permittivity and magnetic permeability and investigate the time-dependent Maxwell equations with self-similar ansatz. This is a first-order hyperbolic partial differential equation system which can conserve non-continuous initial conditions describing electromagnetic shock. Such phenomena may happen in complex materials induced by the planned powerful Extreme Light Infrastructure laser pulses.

Keywords: self-similar solutions, nonlinear Maxwell equations, shock wave

1. Introduction

Wave propagation in nonlinear media is a fascinating field in physics with a large amount of literature [1]. To study such effects diverse nonlinear partial differential equations (PDEs) have to be investigated with various methods. One of the best known nonlinear wave propagation phenomena is the solitary wave, usually based on the nonlinear Schrödinger or sine-Gordon or KdV equations. On the other side there are many more, not so well-known nonlinear wave equations which have delicate properties such as shock-waves, solutions with continuous compact support and so on. Such equations are the various Euler or unconventional heat conduction equations. To investigate if a system has such properties, one of the most powerful analytical tools is to apply the self-similar ansatz which may describe dispersive solutions with reasonable physical interpretation. The validity of such solutions is very wide in continuum mechanics and mostly used to study shock-waves and other fluid dynamical problems [2–4].

In one of our previous studies we investigated the paradox of heat conduction with a new kind of time-dependent Cattaneo heat conduction law [5] and found physically reasonable solutions with compact support. In another analysis we presented three-dimensional analytical results for the Navier–Stokes equations [6]. The properties of the self-similar solution will be analyzed later.

From the four Maxwell field equations combining with the two constitutive relations a linear second-order hyperbolical wave equation can be derived for the field variables. In such cases the constitutive equations contain only linear relations for the electrical permittivity and for the magnetic permeability. The theory of the electromagnetic wave propagation can be found in various textbooks [7].

When discussing the nonlinear Maxwell equation most people mean the non-paraxial nonlinear Schrödinger equation (NNSE) which is derived from the Helmholtz equation including the Kerr media where the relative dielectric permittivity is well described $\epsilon_r = n^2 = n_0^2 + \delta_{\mathrm{NL}}(E_y)$. Here n_0 is the linear contribution to the total refractive index n. In sufficiently slow media, where the characteristic response time of the nonlinearity is much greater than the temporal period of the field oscillations one has $\delta_{\mathrm{NL}} \approx 2n_0^2 n_2 < E_y^2 >$ where n_2 is the Kerr coefficient and <> denotes the time averaging over many optical cycles.

There is a large number of studies available where the NNSE is analytically (or numerically) solved and analyzed. Additional literature can be found in [8].

When ultra short intense laser pulses propagate in a medium then there is an intensity dependence of the group velocity which leads to the phenomena of self-steepening and optical shock-wave formation. It means that the peak of the pulse slows down more than the edge of the pulse, leading to steepening of the trailing edge of the pulse. The envelope becomes steeper and steeper. If the edge becomes infinitely steep, it is said to form an optical shock-wave. Self-steepening has been described by various authors [9–14]. In nonlinear media optical beams can suffer self-trapping where the wave equation is solved with the displacement field of $D = \epsilon_0 E + \eta E^3$ [15, 16].

Most authors consider electromagnetic shock waves in this sense.

We however follow a different way; in our recent study we consider nonlinear, power-law field-dependent electrical permittivity and magnetic permeability and investigate the last two time-dependent Maxwell equations with self-similar ansatz. This is now a first-order hyperbolical nonlinear PDE system which can conserve non-continuous initial conditions describing electromagnetic shock-waves. To our knowledge the question of electromagnetic shock-waves is only briefly mentioned in a well-known physics textbook [17]. Only a small number of publications exist (most of them are by Russian authors) about electromagnetic shock-wave propagations in anisotropic magnetic materials where the direct Maxwell equations [18] are solved. Shock-waves in transition lines are investigated in [19], but unfortunately not using our method.

Direct integration of the Maxwell equations for dielectric resonators is a new research context for future novel particle accelerators [20]. Such effects might happen in complex materials which could be induced by powerful laser pulses which will be available in the planned Extreme Light Infrastructure (ELI).

2. Theory and results

Let's start with the usual four Maxwell equations for the fields:

$$\nabla\cdot\mathbf{D}=\rho,\ \ \nabla\cdot\mathbf{B}=0,\ \ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\ \ \nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J},\quad (1)$$

where **E, B** are the electric and magnetic fields, **D, H** are the electric displacement and magnetizing fields, ρ is the electric charge and **J** is the current density which is zero in insulator media. The closing constitutive relations are

$$\mathbf{D}=\epsilon\mathbf{E},\quad \mathbf{B}=\mu\mathbf{H},\qquad (2)$$

where ϵ is the electrical permittivity and μ is the magnetic permeability. For non-isotropic linear materials ϵ and μ are second order tensors ($D_\alpha = \sum_\beta \epsilon_{\alpha\beta}E_\beta$ and $H_\alpha = \sum_\beta \mu_{\alpha\beta}B_\beta$). For the linear and isotropic materials these are pure real numbers. The most general linear relation for the constitutive equations is however the following

$$D_\alpha(\mathbf{x},t)=\sum_\alpha\int \mathrm{d}^3x'\int \mathrm{d}t'\epsilon_{\alpha,\beta}(\mathbf{x}',t')E_\beta(\mathbf{x}'-\mathbf{x},t-t').\qquad (3)$$

(In equation (3) the $D_\alpha \to B_\alpha$ and $E_\beta \to H_\beta$ interchange is still valid.) This equation means non-locality both in space and time. The latter can be addressed as memory effects, too. The Fourier transform of the electrical permittivity is the frequency dependent dielectric function, which attracts much interest. The crucial symmetry properties can be expressed via the Kramers–Kronig formula [17, 21] which defines the relation between the real and the imaginary part.

Equations of (1)–(2) are enough to derive the usual second-order linear hyperbolic wave-equation for the field variables, which can be found in any electrodynamics textbook [21]. However, this classical calculation is based on a numerical trick, an additional spatial derivation is done, which also means that the first derivatives of the fields are continuous and small. But the original Maxwell equations are of the first order both in time and space. Therefore, the initial conditions do not need to be continuous. This is a crucial point and the main motivation of our analysis.

According to the basic book of Zel'dovich and Raizer [3], which describes the propagation of *large* mechanical disturbances (non-continuous tears, shock-waves) in a medium, the first order hyperbolic Euler and continuity equations have to be applied. These equations also have $f(x-\tilde{c}t)$ traveling wave solutions with a velocity of $\tilde{c}$, which is larger then the propagation of sound. The speed of sound, however, enters the gas dynamic equations. On the other side the propagation velocity of *small* mechanical disturbances can be described via second-order wave equations. In this language we may speak about two different kinds of wave equations, or wave propagation phenomena. The Maxwell equations should be considered for large electromagnetic disturbances and the second order wave equation for the small (e.g. sinusoidal) electromagnetic disturbances. We follow this analogy and apply nonlinear material laws and solve directly the first-order hyperbolic Maxwell equations for propagation.

Maxwell equations in vacuum are linear in the fields of **B** and **E**. Many hundreds of telephone conversations can be propagated in parallel in a single microwave link without any distortion. Another piece of experimental evidence of linearity is the idea of linear superposition. In optics white light is refracted by a prism into the colors of the rainbow and recombined into white light again. There are, of course, circumstances when nonlinear effects occur in magnetic materials, or in crystals responding to intense laser beams, such as frequency doubling.

Our nonlinear Maxwell equation is, however, defined in a completely different way; namely, through the following nonlinear material (or constitutive) equations

$$\mu(\mathbf{H})=a\mathbf{H}^q,\quad \epsilon(\mathbf{E})=b\mathbf{E}^r,\qquad (4)$$

where all the four free parameters (a, b, q, r) are real numbers (for physical reasons $\epsilon(\mathbf{E})\cdot\mu(\mathbf{H})>0$) and the parameters a and b are present to fix the proper physical dimensions. (Such power law dependence of material constants is popular in different flow problems such as in heat propagation [3] where the heat conduction constant can have temperature dependence like $\kappa \sim T^\nu$.) Note, that through these relations we define space and time dependent (dynamical) material equations which are still local in space and time (we neglect now the metamaterials where ϵ and μ could have negative values [22]).

We know from special relativity that the speed of light in a vacuum is the largest available wave propagation which can carry physical information and can be evaluated from electric and magnetic properties as well $c^2 = 1/(\mu_0\epsilon_0)$. (The zero subscript stands for vacuum.) Permeability and permittivity are not fully independent of each other. This formula is slightly modified for any additional media like $c_m^2 = 1/(\mu_0\mu_m\epsilon_0\epsilon_m)$ where the subscript m stands for medium. It is also clear that any stable electromagnetic wave propagation speed in media is always less than the speed of light in a vacuum ($c_m<c$) but for a short time quick particles (usually charged) can propagate faster than the local speed of light, producing

Cherenkov radiation. Therefore, in our calculations we will use the following relation:

$$\mu(H) = aH^q, \quad \epsilon(H) = \frac{1}{c^2 aH^q}. \tag{5}$$

Note that now the propagation speed of the electromagnetic signal has an upper bound, which is c (from now on we will consider one spatial coordinate and neglect the vectorial notation). With this constraint we reduced the number of the four independent parameters to two. In electromagnetic wave propagation the roles of ϵ and μ are symmetric. However, we use this relation because of the existence of $\mathbf{J}$ in the last Maxwell equation. We will see later that with this choice the ordinary differential equation which is obtained from the third Maxwell equation can be integrated and the solutions become more transparent.

For the current density we apply the differential Ohm's law

$$\mathbf{J} = \sigma \mathbf{E}, \tag{6}$$

where σ is the conductance of the media—and can be a second rank tensor in crystals or a highly nonlinear field dependent quality like the permeability or the susceptibility $\sigma = h\mathbf{E}^p$. In a transition-metal oxide it can be a $\sigma \approx (1/E)\sinh(E)$ function [23]. (Our intuition says that only some integers ($\mp$ 1, $\mp$ 2) and some rational numbers ($\mp$ 1/2, $\mp$ 2/3) will be crucially interesting.)

For the sake of simplicity we consider the following one-dimensional wave propagation problem

$$\mathbf{E} = (0, E_y(x,t), 0), \quad \mathbf{H} = (0, 0, H_z(x,t)), \tag{7}$$

which means a linearly polarized electric field in the y direction with x coordinate dependence and a linearly polarized magnetic field in the z direction with x coordinate dependence only. Now the last two Maxwell equations are

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}, \quad -\frac{\partial H_z}{\partial x} = \frac{\partial D_y}{\partial t} + J_y. \tag{8}$$

From basic textbooks [2–4] the form of the one-dimensional self-similar ansatz can be taken

$$T(x,t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right) := t^{-\alpha} f(\eta) \tag{9}$$

where $T(x, t)$ can be an arbitrary variable of a partial differential equation and t means time and x means spatial dependence. The similarity exponents α and β are of primary physical importance since α represents the rate of decay of the magnitude $T(x, t)$, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution as time goes on. Solutions with integer exponents are called self-similar solutions of the first kind (and sometimes can be obtained from dimensional analysis of the problem). The above given ansatz can be generalized considering real and continuous functions $a(t)$ and $b(t)$ instead of t^α and t^β.

The most powerful result of this ansatz is the fundamental or Gaussian solution of the Fourier heat conduction equation (or for Fick's diffusion equation) with $\alpha = \beta = 1/2$. This transformation is based on the assumption that a self-similar solution exists, i.e., every physical parameter preserves its shape during the expansion. Self-similar solutions usually describe the asymptotic behavior of an unbounded or a far-field problem; the time t and the space coordinate x appear only in the combination of $f(x/t^\beta)$. It means that the existence of self-similar variables implies the lack of characteristic length and time scales. These solutions are usually not unique and do not take into account the initial stage of the physical expansion process. These kind of solutions describe the intermediate asymptotics of a problem: they hold when the precise initial conditions are no longer important, but before the system has reached its final steady state. For some systems it can be shown that the self-similar solution fulfills the source type (Dirac delta) initial condition, but not in our next case. They are much simpler than the full solutions and so easier to understand and study in different regions of parameter space. A final reason for studying them is that they are solutions of a system of ordinary differential equations and hence do not suffer the extra inherent numerical problems of the full partial differential equations. In some cases self-similar solutions help to understand diffusion-like properties or the existence of compact supports of the solution.

Applicability of this ansatz is quite wide and comes up in various transport systems [2–6].

For our problem we consider the following ansätze:

$$E_y(x,t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right) := t^{-\alpha} f(\eta);$$
$$H_z(x,t) = t^{-\delta} g\left(\frac{x}{t^\beta}\right) := t^{-\delta} g(\eta). \tag{10}$$

Where α, β, δ are three real numbers which are (at this point of the model) independent of each other. The functions $f(\eta)$ and $g(\eta)$ are the shape functions of the problem.

Combining equation (10) together with equations (4), (6) and inserting it into the original last two Maxwell equations we get the following system:

$$\frac{\partial}{\partial x}[t^{-\alpha} f] = -\frac{\partial}{\partial t}[at^{-\delta(q+1)} g^{q+1}] - \frac{\partial}{\partial x}[t^{-\delta} g]$$
$$= \frac{\partial}{\partial t}[c^{-2} a^{-1} t^{\delta q - \alpha} g^{-q} f] + ht^{-\alpha(p+1)} f^{p+1}. \tag{11}$$

Having done the derivations we arrive at the next ordinary differential equation (ODE) system

$$f' = a(q+1)[\delta g^{q+1} + g^q g' \eta \beta],$$
$$-g' = \frac{1}{ac^2}[(q+1)g^q f + q(q+1)g^{q-1} g' f \eta + (q+1) g^q f' \eta] \tag{12}$$

where prime means derivation with respect to η. Note, that if the following universality relations are held ($\delta = 1$ and $\beta = q + 1$) the first equation is a total derivative and can be integrated resulting

$$f = a(q+1)\eta g^{q+1}. \tag{13}$$

This fixes the connection between the electric and magnetic fields. From the second ODE we get that $\alpha = 1$ and $p = 1$ should be. This means that our media should not have any conductivity for self-similar solutions. Inserting equation (13) into the second equation of (12) we arrive at our final expression

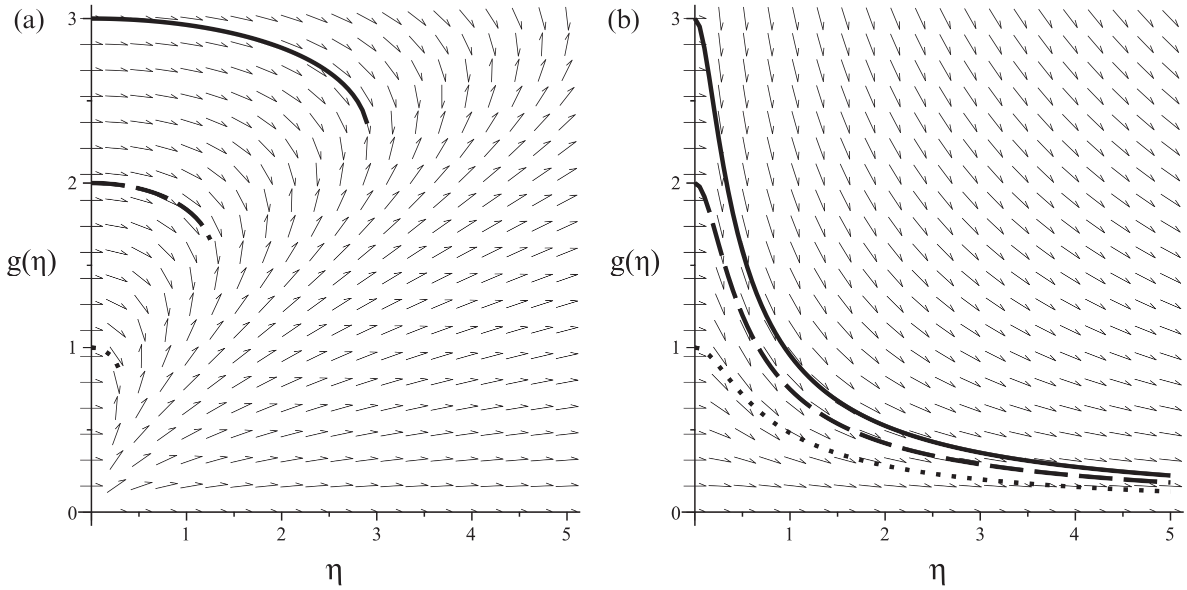

Figure 1. The direction field of equation (14) (*a*) for $q = -2$ and (*b*) for $q = 1/2$. The solid, dashed and dotted lines present numerical solutions $f(0) = 3$, $f(0) = 2$ and $f(0) = 1$ initial conditions for both qs.

$$-g' = \frac{2(q+1)^2 \eta g^{2q+1} + h}{1 + (2q+1)(q+1)^2 \eta^2 g^{2q}} \tag{14}$$

where light velocity c is fixed to unity remaining q and h the final two free parameters. We set h to 0. Note, that now different real q values mean different exponents for magnetic permeability representing different physical material properties and different physics.

For general q only an implicit solution can be given

$$g + g^{2q+1}\eta^2 q^2 + 2g^{2q+1}\eta^2 q + g^{2q+1}\eta^2 - c = 0. \tag{15}$$

For some exponents explicit solutions can be obtained. In general we can investigate the direction field of the ODE which gives us qualitative and global information about the solutions. Note, that (14) is non-autonomous (depending on η) therefore there is no general theorem to study the direction field. A careful analysis for definite q values clearly shows that there are two distinct classes of solutions available.

For $q < -1/2$ there are some solutions with compact supports, otherwise all the solutions are continuous on the whole plain. For $q = -1$ there is an exception: the equation (14) becomes trivial and $g(\eta) = \text{const}$. It is clear from (14) that for $q < -1/2$ the denominator can be zero, therefore a singularity can appear where the first derivative of $g(\eta)$ becomes infinite. This dictates a vertical direction field. If a solution with an initial condition meets this field line than it stops and cannot be continued. This point can be calculated from the denominator. On figure 1(*a*) we present the direction field for $q = -2$. The shock front (or the compact support) is formed on the $g(\eta) = 3^{1/4}\sqrt{\eta}$ which is easy to identify. The compact support of the ODE solution of equation (14) means that the solution of the original PDE system for $E_y(x, t)$ is also compact via the $\eta = x/t^\beta$ in real time and space. The constraint (equation (13)) between f and g also dictates the same compact support for the $H_z(x, t)$ field as well. Outside these time and x coordinate ranges we may fix the values of $E_y(x, t)$ and $H_z(x, t)$ identically to zero which are also solutions of the last two Maxwell equations. In this way we can construct the shock-wave solutions for the original PDEs.

Figure 1(*b*) shows the solution for $q = 1/2$. For $q \geqslant -1/2$ the denominator cannot be zero therefore no infinite derivatives exist. Luckily, the explicit solution can be given, $g(\eta) = \frac{-2c \mp \sqrt{4c + 9\eta^2}}{9c\eta^2}$ which is continuous for every η.

To find physically reasonable solutions we calculate the Poynting vector, which gives us the energy flux (in W/m^2) of an electromagnetic field. Unfortunately, there are two controversial forms of the Poynting vector in material based on the Abraham or the Minkowski formalism; a detailed description can be found in [24]. We use the following form of the Poynting vector:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = t^{-\alpha-\delta} fg = t^{-\alpha-\delta} a(q+1)\eta g^{q+2}. \tag{16}$$

Note that for $q < -2$ the $\int_0^{\text{cut}} S \mathrm{d}\eta$ is finite which is a good result. The spacial integral of the Poynting vector $\sim \int_0^{\text{cut}} \frac{x}{t^\beta} g^{q+2}\left(\frac{x}{t^\beta}\right) \mathrm{d}x$ is finite for all time. However the time integral of $\sim \int_0^{\text{cut}} \frac{x}{t^\beta} g^{q+2}\left(\frac{x}{t^\beta}\right) \mathrm{d}t$ can be problematic for small t and depends on the concrete form of g.

Another method to classify if the solutions are physical would be the total energy of the fields in a finite volume. For linear electrodynamics the energy density is defined as follows: $W = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$. However, even this formula is problematic. There are several nonlinear electrodynamic theories such as those by Born [25] or by Rafelski [26] based on the Lagrangian density where W contains additional terms. Kotel'nikov [27] generalized the Born model and suggested an infinite series of Lorentz and Poincaré-invariant nonlinear versions of the Maxwell equations.

Our approximation to describe the permeability and permittivity equation (4) is just one way to a nonlinear model. Another physically tenable description for the constitutive equations could be a series expansion like $\epsilon(E) = 1 + aE + bE^2 \ldots)$ where the linear term is responsible for the so-called Pockels or electro-optical effect and the quadratic term is for the Kerr effect (a and b are constants to fix the proper dimension). For optically important materials μ = const is the right choice. Nonlinear magnetic properties play a significant role only for plasmas where additional hydrodynamical equations have to be taken into account. Unfortunately, our well-established ansatz equation (10) does not apply directly to such power series.

3. Summary

We introduced a power law magnetic field dependent magnetic permeability and investigated the corresponding nonlinear Maxwell field equations with the self-similar ansatz. If the power law exponent is smaller than minus half then compact, shock-wave like solutions are obtained which might have some importance in laser matter interactions. The work was supported by the Hungarian HELIOS project and by the Hungarian OKTA NK 101438 Grant. This paper is dedicated to my two year old daughter Annabella.

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