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Geometrical origin of chaoticity in the bouncing ball billiard

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ABSTRACT

We present a study of the chaotic behaviour of the bouncing ball billiard. The work is realised on the purpose of finding at least certain causes of separation of the neighbouring trajectories. Having in view the geometrical construction of the system, we report a clear origin of chaoticity of the bouncing ball billiard. By this we claim that in case when the floor is made of arc of circles – in a certain interval of frequencies – one can give semi-analytical estimates on chaotic behaviour.

1. Introduction

Deterministic features of transport have been studied in different problems [1]. These works have also shown that transport may be related to the chaotic aspects of the dynamics [2].

The idea of bouncing ball was studied in different problems where analytical approximations [3] and comprehensive numerical works can also be found [4]. The analytical approximations have been shown the possible evidence of bifurcations while the numerical works presented chaotic regimes of the bouncing ball system.

The *bouncing ball billiard* as a spatial extension of the one dimensional bouncing ball problem has been introduced in [5]. This first work enhances the irregular diffusivity of the system which is similar to certain models of transport [6]. The following work has outlined the spiral modes in the phase space of this problem [7] and also pointed out its relevance on granular matter [8]. Idealised versions where the bounces are performed without loss of energy, i.e. the restitution coefficient is one, and there is no oscillation of the floor may be found in [9].

The *chaoticity* of the sawtooth type of the bouncing ball billiard has been studied in [10], with considerable theoretical background [11]. Considering the problem as a gravitational billiard, aspects on chaotic features are also approached by numerical methods in [12].

The present work focuses on the geometrical origin of chaoticity of the bouncing ball billiard where it is investigated the impact of the curvature of the arcs of circles on the maximal Lyapunov exponent. The derivation shows, that in case of resonance one may give semi-analytical estimates for a lower bound of this exponent.

The study on manifolds for multi-dimensional billiards related to geometric properties is presented in [13]. Certain microscopic aspects of the dynamics in the bouncing ball type models are treated in [14,15]. Possible connections to non-equilibrium phenomena one may find in the works [16,17].

From the practical point of view the reaction of CO with O_2 on Pt surface, which is under thermal excitation, the molecules CO performs diffusive motion on the surface before the reaction would occur [18].

Quasi-deterministic aspects on diffusion may occur in the behaviour of certain species where in the process of food searching one can find a randomness, but there is also a kind of determinism because the animals may have certain remembrances on the places where they found food in the past [19].

The article is organised as follows. In Section 2 we shortly describe the bouncing ball billiard system. The Section 3 makes a presentation of that frequency region where the semi-analytical approaches to some extent are

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possible. Section 4 shows an evaluation which has a semi-empirical and analytical background giving a lower bound for the maximal Lyapunov exponent.

2. The bouncing ball billiard

At this point we make a review of the most important features of the bouncing ball problem. The system studied is a point particle, which bounces on a floor realised of arc of circles. The floor is oscillating with a frequency f corresponding to a circular frequency $\omega = 2\pi f$. The system is presented below. The bouncing ball billiard from the point of view of the diffusion was presented in a comprehensive way in [5,7]. The main conclusions are that the system possesses irregular diffusion, and the principal maximum values for the diffusion occurs at the resonances. These resonances are at the frequencies, where the time of the flight becomes equal or multiple of the period of vibration applied.

The bouncing ball billiard that we study in this paper, with the floor formed by circular scatterers, is depicted in Fig. 1.

The equations of motion of this system are presented as follows: The particle performs a free flight between two collisions in the gravitational field $g \| y$. Consequently, its coordinates (x^-_{n+1}, y^-_{n+1}) and velocities $\left(v^-_{xn+1}, v^-_{yn+1}\right)$ at time t_{n+1} immediately before the $(n+1)$th collision and its coordinates (x^+_n, y^+_n) and velocities $\left(v^+_{xn}, v^+_{yn}\right)$ at time t_n immediately after the nth collision are related by the following equations

$$x^-_{n+1} = x^+_n + v^+_{xn}(t_{n+1} - t_n) \tag{1}$$

$$y^-_{n+1} = y^+_n + v^+_{yn}(t_{n+1} - t_n) - g(t_{n+1} - t_n)^2/2, \tag{2}$$

$$v^-_{x,n+1} = v^+_{xn} \tag{3}$$

$$v^-_{y,n+1} = v^+_{yn} - g(t_{n+1} - t_n). \tag{4}$$

At the collisions the change of the velocities is given by

$$v^+_{\perp n} - v_{ci\perp n} = k(v_{ci\perp n} - v^-_{\perp n}) \tag{5}$$

$$v^+_{\| n} - v_{ci\| n} = \beta\left(v^-_{\| n} - v_{ci\| n}\right), \tag{6}$$

where v_{ci} is the velocity of the corrugated floor. We distinguish between the two different velocity components relative to the normal vector at the surface of the scatterers, where the scatterers are represented by the arcs of the circles forming the floor. $v_\perp$, $v_\|$ and $v_{ci\perp}$, $v_{ci\|}$ is the normal and tangential components of the particle's, respectively the floor's velocity with respect to the surface at the scattering point. Correspondingly, we introduce two different restitution coefficients k and β that are perpendicular, respectively tangential to the normal. These coefficients has their values between zero and one: $k, \beta \in [0,1]$.

As in case of the vertically bouncing ball problem we assume that the floor oscillates sinusoidally, $y_{ci} = -A\sin(\omega t)$, where A and ω are the amplitude respectively the frequency of the vibration, see Fig. 1.

The radius of circles are $R = 15$ mm and the restitution coefficients $k = 0.7$, $\beta = 0.99$, respectively. It is important that the slope of the arcs of the circles is very shallow. The distance between two arc of circles is $d = 2$ mm. By this terms proportional with d^2/R^2 or less are considered terms with second- or higher-order. The shallowness of the arcs implies that the distance between subsequent bounces is small, correspondingly $|x_{n+1} - x_n| < d$, and the terms $(|x_{n+1} - x_n|)^2/R^2$ are similar to d^2/R^2, which is of higher order.

The chaos of the bouncing ball which spatially is a one dimensional system and which is performed on the vertical direction has been discussed in [4]. One of the issues that raised difficulties in the study of the bouncing ball was related to the fact that at certain frequencies the dynamics lead to orbits, which even temporary, becomes stuck to the surface.

In spite of the fact that in two dimensions it is almost impossible to have neighbouring orbits stucked at the same place, the following study will try to avoid that frequency regions where stucking orbits might be possible.

3. General considerations

We discuss the chaoticity of the bouncing ball billiard at the frequency region where the 1/1 resonance holds. The approximation we try to make is semi-empirical and the related considerations are presented below. The first observation we make is, that the time of the flight between subsequent collision at 1/1 resonance is approximately the same. We note this time with t_{fly} and corresponds to that time while the particle makes one bounce and it is close to the time while the floor makes one complete oscillation. Subsequent values of t_{fly} at 1/1 resonance are shown in Table 1.

One can see that these values do not differ too much. They cannot be the same, for instance because the surface is not flat, but they are close to each other and around a specific value.

The other observation is, that the first resonance manifests so that the elongation almost reaches its maximum A,

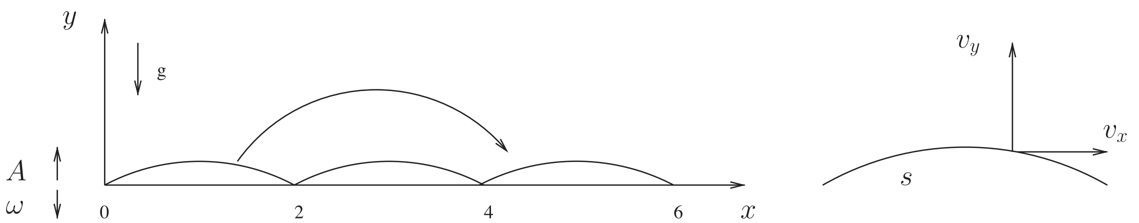

Fig. 1. The bouncing ball billiard. A point particle realises bounces on a vibrating floor consisting of arcs of circles.

Table 1
Time lengths elapsed between subsequent bounces. One can see that they are around a certain value.

Time of flight (s)	Collision no.
0.0183	29
0.0191	30
0.0185	31
0.0190	32
0.0184	33

and the particle meets the floor for almost all cases very close to this height A – see Fig. 2.

4. Geometrical origin of chaoticity

The horizontal velocity component v_x is considered from now on. We are interested in change in difference of the horizontal component of the velocities of two neighbouring trajectories. By this we try to make an estimate of a lower bound of the Lyapunov exponent, which manifests on the v_x direction of the phase space, which would also give a picture on the *horizontal chaoticity* of this problem.

The figure below – Fig. 2 – shows two trajectories, starting from the same place, with slightly different velocity vectors. The picture is at 1/1 resonance, where after a certain transient the bounces are made a little bit above the vertical coordinate A = 0.1 mm which denotes the amplitude.

We assume that at the starting point there is a difference in angles but not in the magnitude of the velocities v_0. This initial deflection in angles we denote by δ. The corresponding difference in initial velocities – on the horizontal projection – we denote by $\Delta v_{x,ini}$. Because of this initial deflection in angle there will be at final arrival a change in horizontal coordinates Δx.

Due to the convex curvature of the arcs characterised by the radius R, the displacement Δx at the arrival on this curvature will cause further deflection in angles after one collision which we denote by δ'. This cause a final difference in the horizontal component of velocities after the first bounce $\Delta v_{x,fin}$.

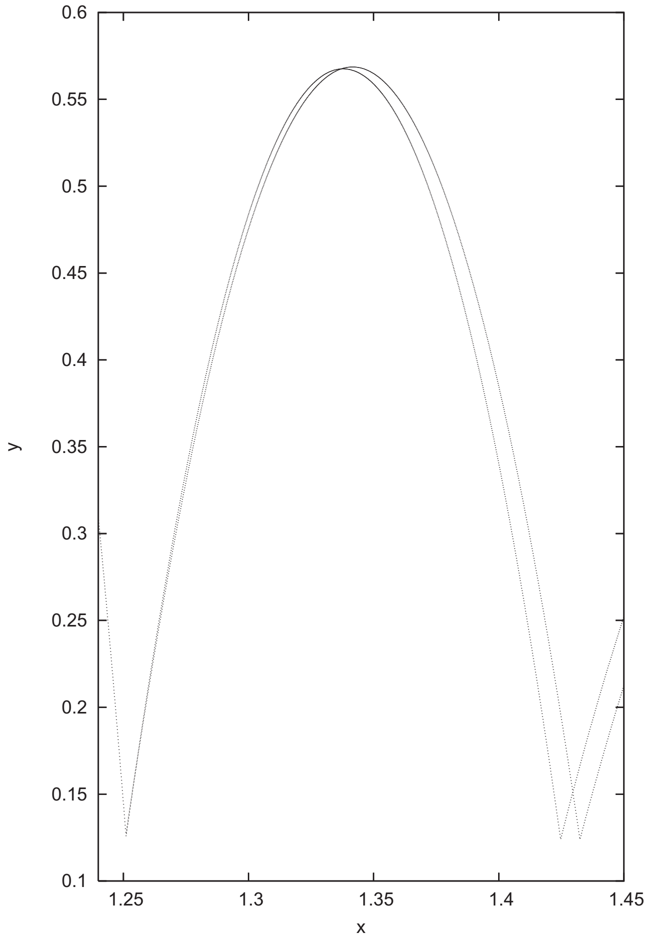

Fig. 2. Typical trajectory of the 1/1 resonance. While the particle arrives at a height close to 0.6, horizontally in general a length around 0.15 is made. The first conclusion is, that the particle in most of the cases will arrive in a steep angle, so the incident angle relative to the normal to the surface is small. The other conclusion is, that quite a number of such bounces occurs while the particle arrives from one arc to another one. The second trajectory is a neighbouring one, caused to a deflection of angles of initial velocities at the starting point.

The rate of exponential separation of the trajectories after such a bounce due to the finite radius R we denote with $\lambda_{vx,R}$

$$\lambda_{vx,R} = \frac{1}{t_{fly}} \ln \frac{|\Delta v_{x,fin}|}{|\Delta v_{x,ini}|} \tag{7}$$

At the launch of the trajectory we consider the magnitude of the velocity v_0, the angles relative to the vertical are $\alpha^{(1)}$ and $\alpha^{(2)}$, the complementary angles are $\alpha_c^{(1)}$ and $\alpha_c^{(2)}$, where as it was mentioned above $\alpha^{(2)} = \alpha^{(1)} + \delta$. Correspondingly

$$\begin{aligned}|\Delta v_{x,ini}| &= \left|v_0 \cos\alpha_c^{(1)} - v_0 \cos\alpha_c^{(2)}\right| = \left|v_0 \sin\alpha^{(1)} - v_0 \sin\alpha^{(2)}\right| \\ &= \left|v_0 \delta \cos\alpha^{(1)}\right| + h.o.t.\end{aligned} \tag{8}$$

where the higher order terms means terms which are proportional with at least the second power of δ. Correspondingly if δ' is the angle between the directions of trajectories after the first bounce

$$|\Delta v_{x,fin}| = |v_0 \delta' \cos\alpha^{(1)'}| + h.o.t. \tag{9}$$

Because there is a free flight in gravitational field and the arcs are with shallow slope practically $\alpha^{(1)'} \simeq \alpha^{(1)}$, very precisely their difference is a second order term. We mention, that $\alpha^{(1)}$ is also small as one can see on the Fig. 2, so its product with δ is also considered a value with second order.[1]

As a result we get for the value $\lambda_{vx,R}$, which has a definite contribution due to the finite value of

$$\lambda_{vx,R} = \frac{1}{t_{fly}} \ln \left|\frac{\delta'}{\delta}\right| + h.o.t. \tag{10}$$

where one can see that the ratio between the deflection of the angles after the bounce and before the bounce counts.

The bounce is presented on Fig. 3.

Based on Fig. 3 we can conclude that the deflection between the two reflected trajectories (δ') one hand is due to the initial deflection δ. For the reflected trajectories there is a contribution due to the curvature. If the point of incidence of the first trajectory is at θ then the point of incidence of the second one is at $\theta + d\theta$. The incident angles relative to the normal differs by $d\theta$ and in addition the reflected angles – on the Fig. 3 – γ_1' and γ_2' has also a difference $d\theta$. Consequently the term $d\theta$ has to be counted twice

$$\delta' = \delta + 2\, d\theta. \tag{11}$$

Inserting this relation in Eq. (10) one gets

$$\lambda_{vx,R} = \frac{1}{t_{fly}} \ln \left|\frac{\delta + 2d\theta}{\delta}\right| + h.o.t. = \frac{1}{t_{fly}} \ln \left|1 + 2\frac{d\theta}{\delta}\right| + h.o.t. \tag{12}$$

At the arriving point on the surface the initial difference δ will cause a displacement Δx. This means that on the arc of the circle the trajectories will arrive at a difference $d\theta \simeq |\Delta x|/R$ as one can see on Fig. 3. Now follows the evaluation of Δx and correspondingly of the $d\theta$.

The length Δx is due to the difference in angles of the velocities, at the starting point. The first one is launched at an angle $\alpha_c^{(1)}$ the other one with an angle $\alpha_c^{(2)} = \alpha_c^{(1)} - \delta$. At the end one of the particles arrives at x_2, the other one at x_1

$$\Delta x = x_1 - x_2 = 2\frac{v_0^2}{g}\left[\cos\alpha_c^{(1)} \sin\alpha_c^{(1)} - \cos\alpha_c^{(2)} \sin\alpha_c^{(2)}\right] \tag{13}$$

where $\alpha_c^{(1)}$ is the angle of the velocity made with the horizontal direction of the first trajectory at the starting point. During the evaluation we make the approximation, that $\sin\delta$ is approximately δ and $\cos\delta \simeq 1$ or the differences are at least second order in δ, and are included in the higher order terms

$$\Delta x = x_1 - x_2 = 2\frac{v_0^2}{g}\delta\left[\cos^2\alpha_c^{(1)} - \sin^2\alpha_c^{(1)}\right] + h.o.t. \tag{14}$$

– here $\alpha^{(1)}$ being the incident angle relative to the vertical. At this point one can see that $\cos^2\alpha_c^{(1)} = \sin^2\alpha^{(1)}$ can be neglected, consequently $d\theta$ yields the following value

$$d\theta \simeq \frac{|\Delta x|}{R} \simeq 2\delta \frac{v_0^2}{gR}\left(\sin^2\alpha_c^{(1)}\right) \tag{15}$$

If we take into account that the time for the flight is $t_{fly} \simeq 2v_0 \sin\alpha_c^{(1)}/g$ then

$$2\frac{d\theta}{\delta} \simeq 4\frac{v_0^2}{gR}\left(\sin^2\alpha_c^{(1)}\right) \simeq \frac{g t_{fly}^2}{R} \tag{16}$$

By this we get for the value $\lambda_{vx,R}$ in leading order

$$\lambda_{vx,R} = \frac{1}{t_{fly}} \ln\left(1 + \frac{g t_{fly}^2}{R}\right) + h.o.t. = \frac{g t_{fly}}{R} + h.o.t. \tag{17}$$

where the logarithm has been expanded, and terms proportional with $1/R^2$ have been also considered as being of higher order.

The average of $\lambda_{vx,R}$ means averaging the expression above. By this the higher order terms vanishes or becomes smaller so they remain of higher order. Consequently it reduces to the average of the time of the flight t_{fly}. Its average is given by the period of oscillations T resulting for $\bar{\lambda}_{vx,R}$ in leading order

$$\bar{\lambda}_{vx,R} \simeq \frac{gT}{R} \simeq \frac{gK}{f}. \tag{18}$$

Because $\bar{\lambda}_{vx,R}$ is a manifestation of the separation of the neighbouring trajectories in the v_x direction due to the geometry, consequently this value can be considered a lower bound for the maximal Lyapunov exponent.

4.1. The case of two periodic orbits

As it is pointed out in the work [3] – with increasing the frequency – bifurcation of the resonant trajectory may be possible. This means that the time of flight consists of a shorter and a longer time alternating one after the other which we denote by $t_{fly,1}$ and $t_{fly,2}$, but they still do not differ too much from each other.[2]

1 Even if a term proportional with α or $\sin\alpha$ would be kept, at the end where the average value is calculated for λ_{vx} or it drops out or it is proved to be of higher order.

2 In general in the case of the bouncing ball billiard to have a considerable difference between $t_{fly,1}$ and $t_{fly,2}$ even it is not possible, or because the dynamics enters in further bifurcations, or simply it crashes to a scenario with lots of sticking orbits.

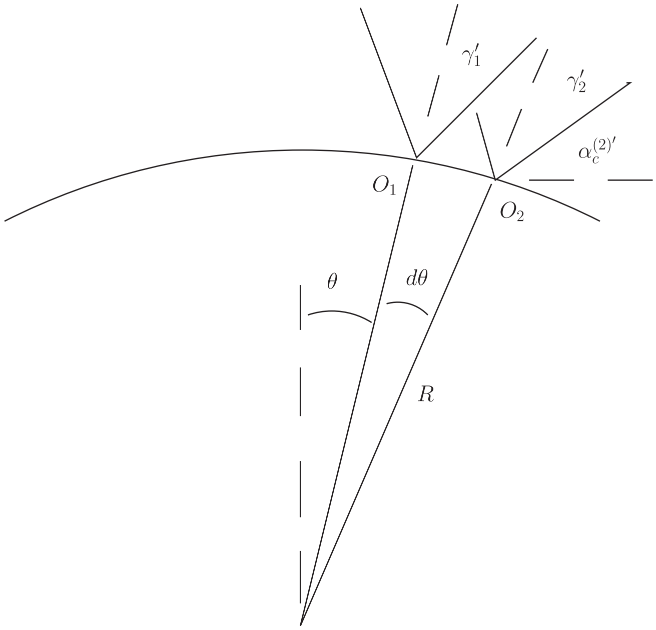

Fig. 3. Illustration of the bounce of neighbouring trajectories on an arc of circle. The figure enhances the displacement of trajectories and their velocities after the bounce due to the geometry of the floor.

In such case the approximation that have been presented previously are still valid and one gets after two consequent flights – one is shorter, one is longer –

$$\lambda_{vx,R} = \frac{1}{t_{fly,1} + t_{fly,2}} \ln\left|\frac{\delta'}{\delta}\frac{\delta''}{\delta'}\right| + h.o.t. \tag{19}$$

The argument of the logarithm can be written as

$$\lambda_{vx,R} = \frac{1}{t_{fly,1} + t_{fly,2}} \ln\left[\left(1 + \frac{g t_{fly,1}^2}{R}\right)\left(1 + \frac{t_{fly,2}^2}{R}\right)\right] + h.o.t. \tag{20}$$

After the expansion to the first order we have

$$\lambda_{vx,R} = \frac{g}{R}\left(\frac{t_{fly,1}^2 + t_{fly,2}^2}{t_{fly,1} + t_{fly,2}}\right) + h.o.t. \tag{21}$$

In the numerator of the second fraction the decompositions are made

$$t_{fly,1(2)} = \frac{t_{fly,1} + t_{fly,2}}{2} \pm \frac{t_{fly,1} - t_{fly,2}}{2}. \tag{22}$$

Finally, we arrive to the relation

$$\lambda_{vx,R} = \frac{g}{R}\frac{t_{fly,1} + t_{fly,2}}{2} + \frac{g}{R}\frac{(t_{fly,1} - t_{fly,2})^2}{2(t_{fly,1} + t_{fly,2})} + h.o.t. \tag{23}$$

This expression can be averaged and the term proportional with $(t_{fly,1} - t_{fly,2})^2$ is still too small and is considered of second order. The average of the last expression yields the value

$$\bar{\lambda}_{vx,R} \simeq \frac{g}{R}\frac{(T_1 + T_2)}{2} \tag{24}$$

where T_1 and T_2 represents the average values of the $t_{fly,1}$, $t_{fly,2}$, respectively. Because even in the case of the two periodic orbit $(T_1 + T_2)/2$ equals an average time flight T' which is the inverse of that frequency f where the dynamics is already bifurcated. This is in fact an interval of frequencies, so the latter formula for two periodic orbits still shows a strong analogy with Eq. (18) and it may be written

$$\bar{\lambda}_{vx,R} \simeq \frac{g}{Rf}. \tag{25}$$

This latter formula shows that the relation (18) may be valid for the full 1/1 resonance and for bifurcated trajectories not too far from it.

5. Conclusions

The problem of chaoticity of the bouncing ball billiard have been put even from the very first years of its appearance. Certain aspects have pointed to the fact that the dynamics may be chaotic, of coarse for a wide interval of the frequency there was a need of numerical simulations. These simulations has shown that the dynamics of the bouncing ball billiard in a wide range of the parameters is chaotic without analysing too much the cause of such a behaviour. In this work we have tried to find at least part of the causes which yields an exponential separation of the neighbouring trajectories. In the case when the dynamics exhibits resonance a certain semi-analytical evaluation can be carried out. The final result shows that the chaoticity is related to the frequency of the vibration, to the gravity and of course to the radius of the arcs of circles of which the floor is made. In principle the chaoticity may have also other causes than the geometrical one, but this might be a further possible problem which may be worth to be investigated.

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