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Investigation of Incompressible Boundary Layers with Viscous Heat Conduction

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Abstract. The incompressible boundary layers are investigated including viscous heat conduction applying the two-dimensional self-similar Ansatz. Analytic solutions are evaluated for the velocity and for the pressure fields. The velocity field can be expressed with the Gaussian and the error functions. Due to the viscous heating term no analytic results are available for the temperature distribution. The parameter dependencies are examined, discussed and shown on the presented figures.

INTRODUCTION

It is obvious that the study of hydrodynamical equations plays an important role in engineering science. The boundary layer theory is a special class of fluid flows. The first pioneering work of this scientific field was made by Prandt [1] who published that using scaling arguments can be derived that about half of the terms of the Naiver-Stokes equations are negligible in boundary layer flows. In 1908, the solutions of the steady-state, incompressible two-dimensional laminar boundary layer equation were published on a semi-infinite plate held parallel to a constant, one-way flow by Blasius [2]. The hydrodynamics of boundary layers has an exhaustive description in the classical textbook of Schlichting [3] and the recent applications in engineering science are discussed by Hori [4]. Numerous researchers focused on the mathematical properties of the corresponding partial differential equations (PDEs). Without completeness, we list some of the available mathematical studies. Libby and Fox [5] gave some solutions using perturbation method. Ma and Hui [6] introduced similarity solution to the boundary layer problems. Burde [7, 8, 9] published numerous explicit analytic solutions in the nineties. Weidman [10] presented solutions for boundary layers with additional cross flows. Ludlow and coworkers [11] obtained and examined solutions with similarity methods as well.

The steady-state boundary layer flow equations for non-Newtonian fluids were analyzed by Bognár [12] and she presented self-similar results. After that, it was generalized [13, 14, 15, 16, 17, 18, 19], and the steady-state heat conduction mechanism was included. In our former publications, we investigated the full two-dimensional viscous flows coupled to the heat conduction equation, which are three different kinds of Rayleigh-Bénard heat conduction problems [20, 21, 22]. We can say that a simplified version of the Rayleigh-Bénard problem is a kind of boundary layer equation with heat conduction.

In the present study, the Sedov type self-similar Ansatz [23, 24] will be applied to transform the original partial differential equation (PDE) systems to a non-linear ordinary differential equation (ODE) systems. The ODE systems can be solved giving analytical solutions for the velocity, pressure, and temperature fields with quadrature. Due to our best knowledge, no analytical results exist by the present study for any type of time-dependent boundary layer equations including heat conduction.

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THE GOVERNING EQUATIONS

We investigate the following PDE system:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial p}{\partial y} = 0, \tag{2}$$

$$\rho_\infty \frac{\partial u}{\partial t} + \rho_\infty \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial x} \right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}, \tag{3}$$

$$\rho_\infty c_p \frac{\partial T}{\partial t} + \rho_\infty c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} - a \left(\frac{\partial u}{\partial y} \right)^2, \tag{4}$$

where the variables are the two velocities components $u(x, y, t), v(x, y, t)$ of the fluid, the pressure $p(x, y, t)$, and the temperature $T(x, y, t)$. The $\rho_\infty, c_p, \mu, \kappa, a$ are the additional physical parameters, the fluid density at asymptotic distances and times, the heat capacity at fixed pressure, the kinematic viscosity, the thermal diffusivity and the strength of the viscous heating term, respectively. We use the following form of the self-similar Ansatz:

$$\begin{aligned} u(x, y, t) &= t^{-\alpha} f(\eta), \qquad v(x, y, t) = t^{-\delta} g(\eta), \\ T(x, y, t) &= t^{-\gamma} h(\eta), \qquad p(x, y, t) = t^{-\epsilon} i(\eta), \end{aligned} \tag{5}$$

where the new argument $\eta = \frac{x+y}{t^\beta}$ means the shape functions. All the exponents $\alpha, \beta, \gamma, \delta$ are real numbers.

The shape functions f, g, h and i are continuous functions with existing first and second continuous derivatives and will be evaluated later on. The logic, the physical and geometrical interpretation of the Ansatz were exhaustively analyzed in all our former studies [20, 21, 22]. It is necessary to remark that $\alpha, \delta, \gamma, \epsilon$ are responsible for the rate of decay and β is for the rate of spreading of the corresponding dynamical variable for positive exponents. In our case, the numerical values of the exponents are the following:

$$\alpha = \beta = \delta = 1/2, \qquad \epsilon = 1, \qquad \gamma = 1. \tag{6}$$

The one-half values refer to the regular Fourier heat conduction (or Fick's diffusion) process. The derived ODE system reads

$$f' + g' = 0, \tag{7}$$

$$i' = 0, \tag{8}$$

$$\rho_\infty \left(-\frac{f}{2} - \frac{f'\eta}{2} \right) + \rho_\infty (f f' + g f') = \mu f'' - i', \tag{9}$$

$$\rho_\infty c_p \left(\gamma h - \frac{h'\eta}{2} \right) + \rho_\infty c_p (f h' + g h') = \kappa h'' - a(f')^2, \tag{10}$$

here prime means derivation in respect to the reduced variable η. After some straightforward algebraic manipulation we arrive to the next separate ODE for the velocity shape function $f(\eta)$

$$\frac{1}{\rho_\infty}(\mu f' + c_2 \eta + c_3) + \frac{f \cdot \eta}{2} - c_1 f = 0, \tag{11}$$

with the analytic solution of

$$f = \left(-\frac{2 c_2 \mu e^{\left[\frac{\rho_\infty \eta^2}{4\mu} - \frac{c_1 \eta \rho_\infty}{\mu} \right]}}{\mu \rho_\infty} - \frac{\sqrt{\pi} e^{\left\{ -\frac{c_1^2 \rho_\infty}{\mu} \right\}} erf \left\{ \frac{1}{2} \sqrt{-\frac{\rho_\infty}{\mu}} + \frac{c_1 \rho_\infty}{\mu \sqrt{-\frac{\rho_\infty}{\mu}}} \right\}}{\sqrt{-\mu \rho_\infty}} (2 c_1 c_2 + c_3) + c_4 \right) \times e^{\frac{\eta(-\eta + 4 c_1)\rho_\infty}{4\mu}} \tag{12}$$

where erf means the usual error function [25]. Note, that for the positive real constants ρ_∞, μ the complex quantity $\sqrt{-\rho_\infty/\mu}$ appears in the argument of the error functions and as a complex multiplicative prefactor simultaneously makes the final result a pure real function. The crucial parameter is the ρ_∞/μ ratio, if this is larger than unity then the function tends to a sharp Gaussian. Figure (1) shows the general velocity shape function (12) for various parameter sets. The choice of these parameters are arbitrary, however we try to create the most general and most informative figures.

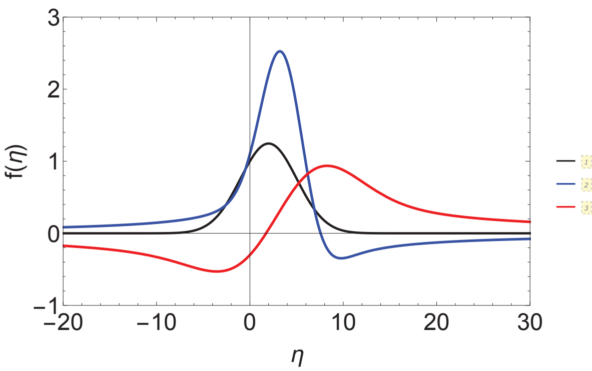

FIGURE 1. The graphs of the velocity shape function $f(\eta)$ in Eq. (12) for three different parameter sets $(c_{1,2,3,4}, \mu, \rho_\infty)$. The black, blue and red lines are for $(1, 0, 1, 0.3, 4.1, 0.9)$, $(2, -1, 2, 0.3, 1.5, 1)$ and $(2, 2, 0.3, 0.3, 3, 3)$, respectively.

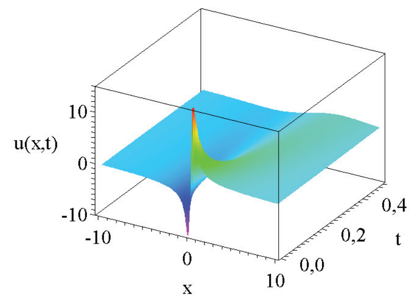

FIGURE 2. The velocity distribution function $u(x, y = 0, t) = \frac{1}{t^{1/2}} f(\eta)$ for the second parameter set presented on the previous figure.

Figure (2) presents the velocity distribution function. Note, very sharp peak in the origin and the extreme quick time decay along the time axis.

With the knowledge of the velocity shape function the ODE for the temperature shape function can be determined and reads as follows:

$$\kappa h' + \rho_\infty c_p h' \left(\frac{\eta}{2} - c_1\right) + \rho_\infty c_p h - a(f')^2 = 0. \tag{13}$$

Unfortunately, (even for the simplest case where the velocity shape function $f(\eta)$ is just the Gaussian function) there are no closed form analytic solutions available. Therefor we have to fix the physical parameters $(\rho_\infty, c_p, \kappa)$ and the integration constants to some typical values and numerical integration procedures have to be applied. Figure (3) presents the shape function for the three different a strength parameters. Note, that larger the parameter the larger the finally reached value of the function. Figure (4) we present the temperature distribution function for $a = 1$. The function remains similar to the shape function.

For the sake of completeness we present the solutions for the pressure field as well. The ODE of the shape function is trivial: $i' = 0$ with the solution of: $i = c_4$. Therefore the final pressure distribution reads: $p(x, y, t) = t^{-\epsilon} \cdot i(x, y, t) = \frac{c_4}{t}$, which means that the pressure is constant in the entire space at a given time point, but has a quicker time decay than the velocity field.

SUMMARY

We investigated the incompressible time-dependent boundary flow equations with additional viscous heat conduction mechanism with the self-similar Ansatz. The velocity field is analytic and can be expressed with the error functions (in some special cases with Gaussian functions). However, the temperature field - due to the viscous heating - cannot be described with analytic means. Further work is in progress to extend his model eg. with inclusion of non-newtonian viscosity effects.

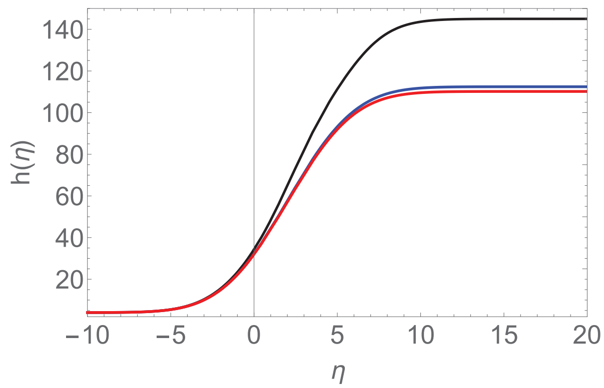

FIGURE 3. The graphs of the temperature shape function $h(\eta)$ in Eq. (13) for three different strength parameter a The black, blue and read lines are for $a = 10, 1$ and 0 respectively. All other parameters are the same as the second set on Figure 1.

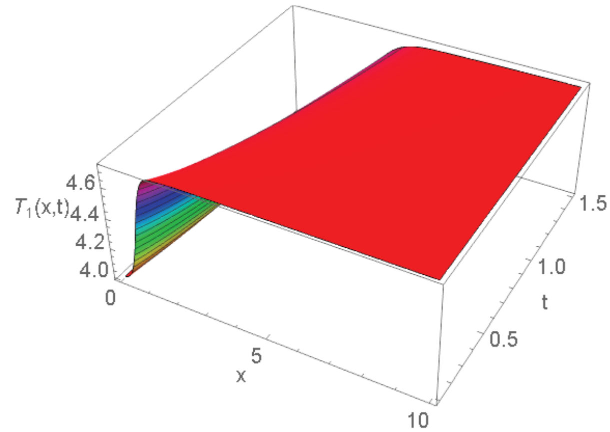

FIGURE 4. The Temperature distribution function $T(x, y = 0, t) = \frac{1}{t^1} h(\eta)$ for $a = 1$. All other parameters are the same as the second set on Figure 1.

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