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SELF-SIMILAR SOLUTIONS OF THE TWO DIMENSIONAL HEAT DIFFUSION EQUATION FOR INFINITE HORIZON

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Abstract. In this article the two dimensional heat diffusion equation is studied. We consider a system for infinite horizon, the heat can diffuse without spatial constraint. In case of polar coordinates one have a radial and an angular part of the spatial variation. With an appropriate self-similar transformation, we arrive to an ordinary differential equation. The equation admits a countable set of solutions which can be obtained by an algebraic method. These solutions decay in space and in time, the latter is a power law decay with different exponents for different solutions. The diffusion equation with a source term is also discussed by the self-similar method.

Key words: heat diffusion, mass diffusion, partial differential equation, self-similar solution.

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1. INTRODUCTION

The transfer of heat is one of the most common phenomenon which surround us, which determines fundamental processes from technical applications to living systems. The study of heat conduction dates back among the others to very early works of Fourier, and by further developments more complex results have been found [1, 2]. In related context, the phenomena of spreading or slow penetration is also important in a numerous aspects of life, in nature or engineering [3–5]. Regarding mathematical analysis of heat and mass diffusion, thorough studies have been realized in [6], and detailed works regarding different cases, with boundary conditions on compact domain and explicit applications can be found in [7, 8].

The fundamental solution of the one dimensional mass or heat diffusion equation for infinite horizon, when a finite amount of mass or energy can diffuse without special finite constraints, is the Gaussian solution. This solution have been discussed in the references mentioned above [3, 6], and also in other works like [9].

The one dimensional problem of mass or heat diffusion have been studied with certain methods, and Bluman and Cole have arrived to results presented in [10]. The method used, recovers the Gaussian form and related expressions which show that in the exponent not only x^2/t on some power is possible, and one can also find some slight generalizations of this argument.

The problem for infinite horizon have got a new development, with the works [11, 12], where explicit solutions of the one dimensional case are given beyond Gaussian. The countable set of even and odd solutions of the diffusion equation for spatially extended one dimensional system are presented in [13].

In this study we focus on the two dimensional heat diffusion equation with infinite horizon. Regarding the case of finite horizon, relevant engineering aspects are discussed in [8].

On general grounds heat conduction occur in different mediums with corresponding heat transfer [14, 15]. The heat diffusion equation has also similarities to the mass or particle diffusion [16]. An important application of the two dimensional diffusive case is related to agar diffusion, where the spreading of certain substances is studied [17, 18]. Assuming that a substance can diffuse in agar, its spreading may cause suppression of bacterial growth in case of antibiotics [19, 20]. The method can provide relatively fast result on the effects of antimicrobial agents. The area where these substances acts form an inhibition zone. The size of this domain shows the effectiveness of the substance on microbes. The agar diffusion method can be used both for synthetic and extracted – *e.g.* plant extracts – antimicrobial substances. Diffusion with convection occur in lungs during respiration [21, 22]. The features of diffusion with reaction, with possible cross effects, have been studied in [23, 24]. Regarding biological aspects of diffusion certain number of studies can be found. The features of Brownian motion are described and mathematical models of it has been presented in [25]. Discrete models of diffusion have wide application area and they are used in different scientific fields [26–28]. The two dimensional diffusion problem has important application in food industry [29]. Interesting applications related to heat transfer one may find in area of micro-thermoelectric cooling [30], thermal management of batteries [31, 32], and in case of nanofluids [34–36]. Further generalizations are the diffusion equations, where one may find concentration dependence of diffusion coefficient [37–39]. Numerical methods for solving the equations related to diffusion or heat transfer in different problems are presented in [40]. Self-similar propagation of optical beams has been studied in [41].

The heat diffusion equation in general reads as follows

$$\frac{\partial T}{\partial t} = D\nabla^2 T, \tag{1}$$

where D is the heat diffusion coefficient. In this work, we suppose that the diffusion coefficient is constant in the domain, which is studied.

2. SELF-SIMILAR ANALYSIS

The heat equation in one spatial dimension have been analyzed in the considerable previous work [10], where the idea of a specific change of variables is also mentioned. The work presents the basic idea of the self-similar transformation, and for the diffusion problem a Boltzmann type of change of variables is used: $x/\sqrt{t}$.

Making a step further, a more general transformation have been applied to this one dimensional problem by Mátyás and Barna, which is the following:

$$T(x,t) = \frac{1}{t^{\alpha}} f\left(\frac{x}{t^{\beta}}\right). \tag{2}$$

Applying this transformation, it turned out, that $\beta = 1/2$ and α can be an arbitrary value [11]. For each α one obtains a specific solution. The sets of countable solutions related to integer or half integer values of α have certain symmetries – being odd or even –, and they have been presented in [13].

In case of two dimensions the equation (1) has the form

$$\frac{\partial T}{\partial t} = D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + D\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}. \tag{3}$$

If we suppose spatial isotropy, when there is no angle dependence, then only the radial terms remain

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial r^2} + D\frac{1}{r}\frac{\partial T}{\partial r}. \tag{4}$$

One can apply in this case a similar transformation

$$T(r,t) = \frac{1}{t^{\alpha}} f\left(\frac{r}{\sqrt{t}}\right). \tag{5}$$

In the following we use the variable

$$\eta = r/\sqrt{t}. \tag{6}$$

Finally we obtain the following differential equation

$$-\alpha f - \frac{1}{2}\,\eta\,\frac{df}{d\eta} = D\,\frac{1}{\eta}\,\frac{df}{d\eta} + D\frac{d^2 f}{d\eta^2}. \tag{7}$$

The general solutions of this equation can be expressed in terms of Kummer functions as it is mentioned in [43].

3. SOLUTIONS OF THE TRANSFORMED EQUATION

However the sum of the linear combination of Kummer functions can be quite a complicate expression, in case the $\alpha = 1$ we expect that a classical solution will be recovered. Inserting in equation (7) the function

$$f(\eta) = e^{-\frac{\eta^2}{4D}}, \tag{8}$$

one finds, that it fulfills the equation (7). Consequently we get the generic solution for two dimension

$$T(r,t) = K\frac{1}{t}e^{-\frac{r^2}{4Dt}}, \tag{9}$$

where K is a constant, which should be fixed, by the initial or boundary conditions of the problem. This function one can see in Fig. 1.

We mention, that we give the solutions with the diffusion coefficient in the formula, as it is for example in (9), so that they can be used in or compared with experimental works, in a relatively direct way. Regarding our graphical representations we take $D = 1$. An alternative possibility is to use the variable $t' = Dt$; in this case the solution can be written as $T(r,t) = K'\frac{1}{t'}e^{-\frac{r^2}{4t'}}$, with $K' = KD$.

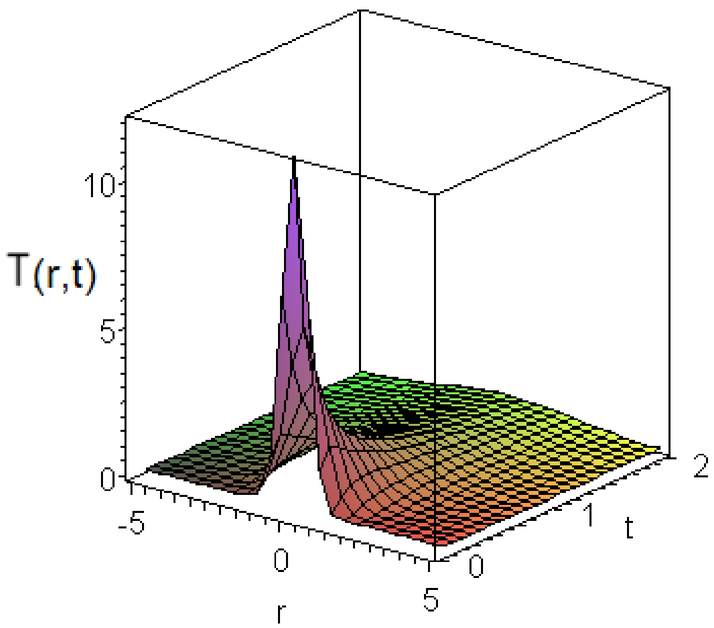

Fig. 1 – Solution of the radial heat equation for $\alpha = 1$, with $K = 1$, and D has unit value.

In case $\alpha = 2$ we have the conjecture, that the solution of the equation can be searched in the following form

$$f(\eta) = e^{-\frac{\eta^2}{4D}} \left(K_0 + K_1\eta^2\right). \tag{10}$$

This yields for the derivative of the function

$$\frac{df}{d\eta} = -e^{-\frac{\eta^2}{4D}} \frac{\eta}{2D}(K_0 + K_1\eta^2) + e^{-\frac{\eta^2}{4D}}\, 2\,\eta\, K_1. \tag{11}$$

We insert this expression in the original equation (7) and it turns out, that the form (10) is solution of the equation if

$$K_1 = -\frac{K_0}{4D}. \tag{12}$$

Correspondingly, the form of the function f for $\alpha = 2$ is given by the following expression

$$f(\eta) = e^{-\frac{\eta^2}{4D}}(K_0 + K_1\eta^2) = K_0\, e^{-\frac{\eta^2}{4D}} \left(1 - \frac{1}{4D}\eta^2\right). \tag{13}$$

The complete solution related to this shape function is

$$T(r,t) = K_0 \frac{1}{t^2} e^{-\frac{r^2}{4Dt}} \left(1 - \frac{1}{4D}\frac{r^2}{t}\right). \tag{14}$$

This function one can see on Fig. 2.

In case $\alpha = 3$ we have the consideration, that the function f has the form

$$f(\eta) = e^{-\frac{\eta^2}{4D}} \left(K_0 + K_1\eta^2 + K_2\eta^4\right). \tag{15}$$

The derivative of this function is

$$f' = \frac{df}{d\eta} = -e^{-\frac{\eta^2}{4D}} \frac{\eta}{2D}(K_0 + K_1\eta^2 + K_2\eta^4) + e^{-\frac{\eta^2}{4D}}\, 2\,(K_1\,\eta + 2K_2\eta^3). \tag{16}$$

Inserting this expression in the original equation (7), we obtain the following relations for the coefficients

$$K_1 = -\frac{K_0}{2D}, \tag{17}$$

and

$$K_2 = -\frac{K_1}{16D} = \frac{K_0}{32D^2}. \tag{18}$$

These relations imply, the following form for the shape function

$$f(\eta) = K_0 e^{-\frac{\eta^2}{4D}} \left(1 - \frac{1}{2D}\eta^2 + \frac{1}{32D^2}\eta^4\right). \tag{19}$$

The complete solution in terms of time and space variables reads as follows

$$T(x,t) = K_0 \frac{1}{t^3} e^{-\frac{r^2}{4Dt}} \left(1 - \frac{1}{2D}\frac{r^2}{t} + \frac{1}{32D^2}\frac{r^4}{t^2}\right). \tag{20}$$

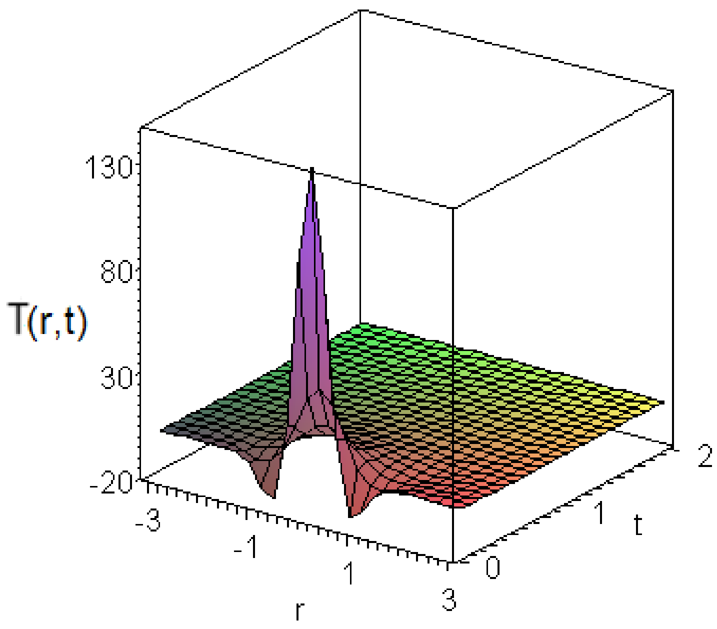

Fig. 2 – Solution of the two dimensional heat equation for $\alpha = 2$, $K_0 = 1$, D set to unity.

The form of $T(r,t)$ on a three dimensional plot one can see on Fig. 3.

For $\alpha = 4$ we look for the following series expansion for f

$$f(\eta) = e^{-\frac{\eta^2}{4D}} \left(K_0 + K_1\eta^2 + K_2\eta^4 + K_3\eta^6\right). \tag{21}$$

Inserting this form into the equation (7), we arrive to the following ratio of the coefficients

$$\frac{K_1}{K_0} = -\frac{3}{4}, \tag{22}$$

and

$$\frac{K_2}{K_1} = -\frac{1}{8}, \tag{23}$$

and

$$\frac{K_3}{K_2} = -\frac{1}{36}. \tag{24}$$

Finally we get for the form of the shape function

$$f(\eta) = K_0 e^{-\frac{\eta^2}{4D}} \left(1 - \frac{3}{4D}\eta^2 + \frac{3}{32D^2}\eta^4 - \frac{1}{384D^3}\eta^6\right). \tag{25}$$

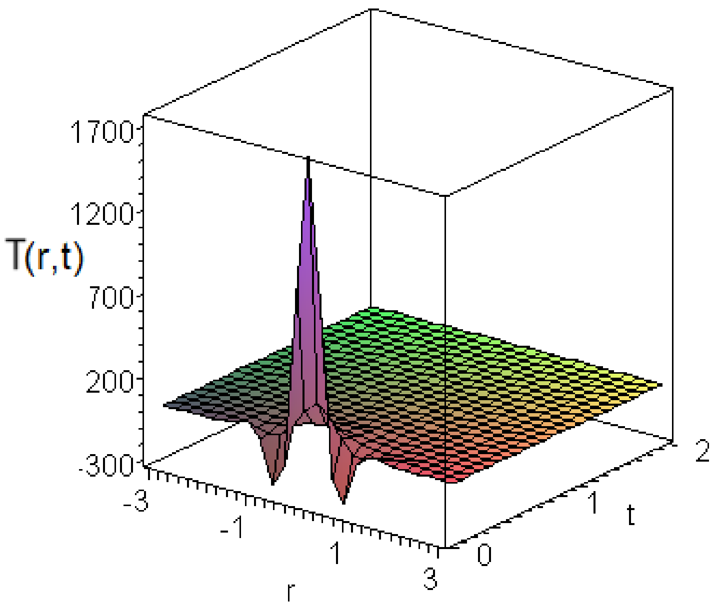

Fig. 3 – Solution of the heat equation for $\alpha = 3$, $K_0 = 1$, D has unit value.

The complete solution $T(x,t)$ of diffusion equation is

$$T(r,t) = K_0 \frac{1}{t^4} e^{-\frac{r^2}{4Dt}} \left(1 - \frac{3}{4D}\frac{r^2}{t} + \frac{3}{32D^2}\frac{r^4}{t^2} - \frac{1}{384D^3}\frac{r^6}{t^3}\right). \tag{26}$$

This solution of $T(r,t)$ one can see on Fig. 4.

The solutions above are symmetric relative to the space variable r, with the property, that at $+\infty$ and $-\infty$ these solutions vanishes.

As we can see, each function fulfill the equation of diffusion, consequently any linear combination of it is a solution of the equation. In fact more linear combinations are possible, if one take two or more specific solutions presented above. Such constructions presents mixed distributions of the physical parameter which diffuses. For instance the solution for $\alpha = 1$ and $\alpha = 3$ with the following weights can be written:

$$T(r,t) = \frac{1}{t} e^{-\frac{r^2}{4Dt}} - 0.1\frac{1}{t^3} e^{-\frac{r^2}{4Dt}} \left(1 - \frac{1}{2D}\frac{r^2}{t} + \frac{1}{32D^2}\frac{r^4}{t^2}\right). \tag{27}$$

The form of the linear combination (27) can be seen in Fig. 5.

Regarding initial conditions we take the following initial distribution at time $t = 0$

$$g(x_0, y_0) = Heaviside(9 - x_0^2) \cdot Heaviside(9 - y_0^2). \tag{28}$$

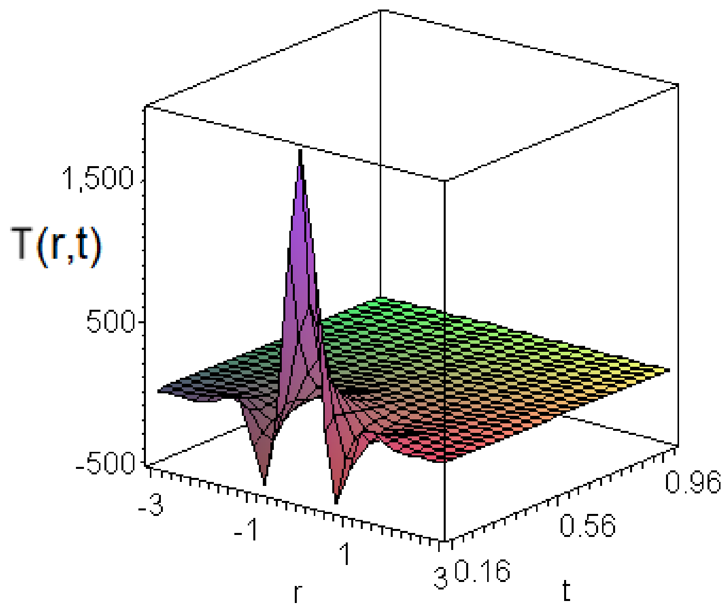

Fig. 4 – Solution of the heat equation for $\alpha = 4$, with $K_0 = 1$, $D = 1$.

The solution in this case can be found from the convolution integral [44]

$$T(x,y,t) = \frac{1}{4\pi t}\int_{\infty}^{\infty} Heaviside(9-x_0^2)\cdot Heaviside(9-y_0^2)e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4t}}dx_0dy_0. \tag{29}$$

Here for simplicity we have taken $D = 1$. Evaluating this integral one gets

$$T(x,y,t) = \frac{1}{4}\left[erf\left(\frac{3}{2\sqrt{t}}+\frac{x}{2\sqrt{t}}\right)erf\left(\frac{3}{2\sqrt{t}}+\frac{y}{2\sqrt{t}}\right) - erf\left(\frac{3}{2\sqrt{t}}+\frac{x}{2\sqrt{t}}\right)erf\left(-\frac{3}{2\sqrt{t}}+\frac{y}{2\sqrt{t}}\right)\right.$$
$$\left. - erf\left(-\frac{3}{2\sqrt{t}}+\frac{x}{2\sqrt{t}}\right)erf\left(\frac{3}{2\sqrt{t}}+\frac{y}{2\sqrt{t}}\right) + erf\left(-\frac{3}{2\sqrt{t}}+\frac{x}{2\sqrt{t}}\right)erf\left(-\frac{3}{2\sqrt{t}}+\frac{y}{2\sqrt{t}}\right)\right]. \tag{30}$$

The form of this function for different times one can see on Fig. 6.

4. DISCUSSION: PROPERTIES OF NEW SOLUTIONS

The new solution implies new descriptions of mass or heat distribution, consequently the integrals on full space of the new solutions are important. If one would study mass diffusion, the integral on the whole space in case of concentration means the total mass in the system which diffuses. In case of $\alpha = 1$ the shape function f,

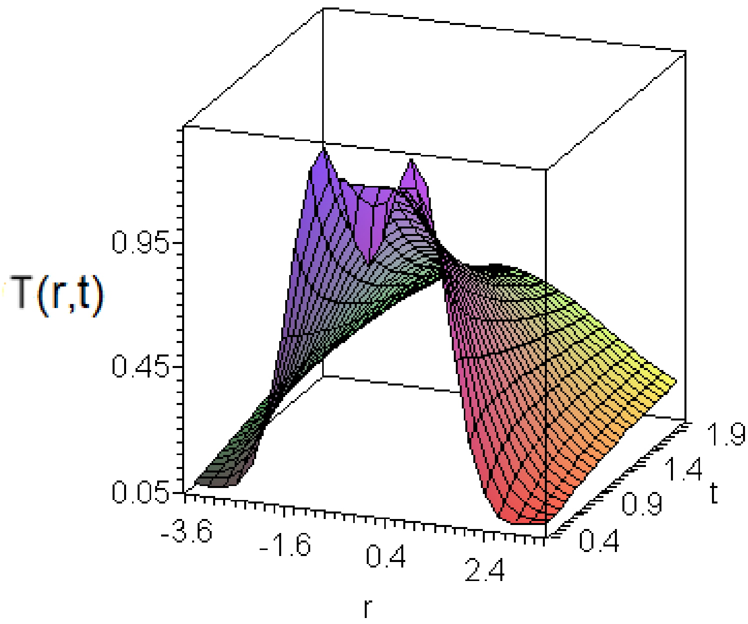

Fig. 5 – Linear combination of solutions related to $\alpha = 1$ and $\alpha = 3$.

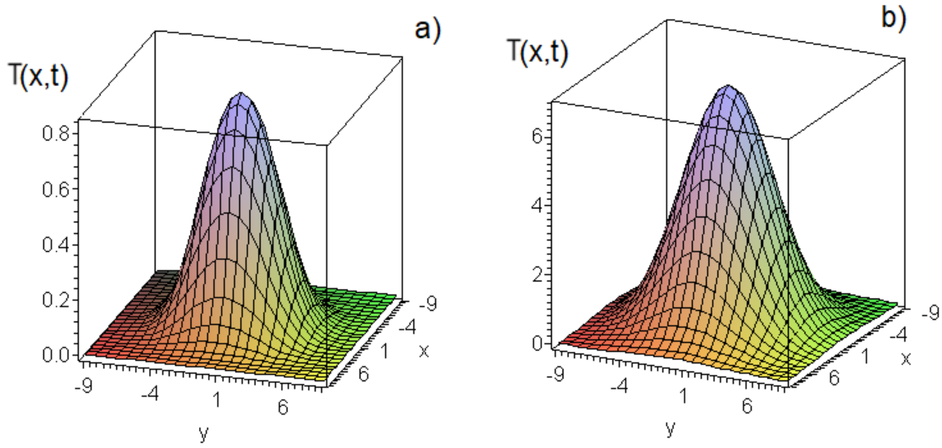

Fig. 6 – The solution with given initial condition, related to time $t = 1.5$ a), and $t = 3.5$ b).

with the argument $\eta = \sqrt{x^2+y^2}/\sqrt{t} = r/\sqrt{t}$ has the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\eta)dxdy = \int_{0}^{\infty} f(\eta)2\pi r dr = \int e^{-\frac{r^2}{4Dt}} 2\pi r dr = 4\pi Dt. \qquad (31)$$

This means, that for the value of $T(r,t)$

$$\int_0^\infty T(r,t) 2\pi r dr = \int K \frac{1}{t} e^{-\frac{r^2}{4Dt}} 2\pi r dr = K 4\pi D. \tag{32}$$

If $T(r,t)$ stand for the concentration, the $K4\pi D$ is the total mass in the system which can be fixed, by the constant K, as it will discussed further later.

The next relevant value is $\alpha = 2$. The integral of the shape function in itself

$$\int_{-\infty}^\infty f(\eta) d\eta = e^{\frac{-\eta^2}{4D}} \left(1 - \frac{\eta^2}{4D}\right) = \sqrt{\pi D}. \tag{33}$$

Taking into account, that from physical point of view η depends on two spatial variables $\eta = \sqrt{x^2+y^2}/\sqrt{t} = r/\sqrt{t}$, the integral on the whole two dimensional space turns out to be

$$\int_0^\infty f(\eta) 2\pi r dr = \int_0^\infty e^{\frac{-\eta^2}{4D}} \left(1 - \frac{\eta^2}{4D}\right) 2\pi r dr = \lim_{r\to\infty} \pi r^2 e^{\frac{-r^2}{4Dt}} = 0. \tag{34}$$

Consequently, we arrive to the important information

$$\int_0^\infty T(r,t) 2\pi r dr = \int_0^\infty \frac{1}{t^2} e^{\frac{-r^2}{4Dt}} \left(1 - \frac{r^2}{4Dt}\right) 2\pi r dr = 0, \tag{35}$$

for any finite time. This fact means, that the solution corresponding to $\alpha = 2$, which follows the Gaussian ($\alpha = 1$), has integral zero on the whole two dimensional space.

For $\alpha = 3$ we have for the integral of the shape function

$$\int_{-\infty}^\infty f(\eta) d\eta = e^{\frac{-\eta^2}{4D}} \left(1 - \frac{\eta^2}{2D} + \frac{\eta^4}{32D}\right) = \frac{3}{4}\sqrt{\pi D}. \tag{36}$$

In two dimensions

$$\int_0^\infty f(\eta) 2\pi r dr = \int_0^\infty e^{\frac{-\eta^2}{4D}} \left(1 - \frac{\eta^2}{2D} + \frac{\eta^4}{32D}\right) 2\pi r dr = \lim_{r\to\infty} \pi \frac{1}{8} \frac{r^2 e^{\frac{-r^2}{4Dt}}(8Dt - r^2)}{Dt} = 0. \tag{37}$$

and

$$\int_0^\infty T(r,t) 2\pi r dr = \int_0^\infty \frac{1}{t^3} e^{\frac{-r^2}{4Dt}} \left(1 - \frac{r^2}{2Dt} + \frac{r^4}{32Dt}\right) 2\pi r dr = 0. \tag{38}$$

This means that a linear combination of the type (27) reflects a certain local distribution of heat, although the relevant cumulative part is determined by (32).

5. THE TWO-DIMENSIONAL RADIAL EQUATION WITH CONSTANT SOURCE TERM

The diffusion equation with different source terms has been studied especially for one dimension [39]. If one consider complex diffusion coefficient with possible further terms in the equation one may arrive to the Schrödinger or a family of similar equations [45, 46].

At this point, we add a constant source term to the two dimensional heat diffusion equation in its radial form

$$\frac{\partial T}{\partial t} = D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + D\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + q. \tag{39}$$

where q indicates a constant source.

Applying the transformation (2) one arrive to the following ODE

$$-\alpha t^{-\alpha-1} f - \beta t^{-\alpha-1}\eta\frac{\partial f}{\partial \eta} = D\frac{1}{\eta}t^{-\alpha-2\beta}\frac{\partial f}{\partial \eta} + Dt^{-\alpha-2\beta}\frac{\partial^2 f}{\partial \eta^2} + q. \tag{40}$$

All the terms has the same long time decay if

$$-\alpha - 1 = 0 \tag{41}$$

$$-\alpha - 2\beta = 0. \tag{42}$$

The explicit values for α and β are

$$\alpha = -1 \tag{43}$$

$$\beta = \frac{1}{2}. \tag{44}$$

The final differential equation for f as a function of $\eta = x/\sqrt{t}$ is

$$f - \frac{1}{2}\eta\frac{\partial f}{\partial \eta} = D\frac{1}{\eta}\frac{\partial f}{\partial \eta} + D\frac{\partial^2 f}{\partial \eta^2} + q. \tag{45}$$

Introducing the function $h(\eta) = f(\eta) - q$, we arrive to the equation

$$h - \frac{1}{2}\eta\frac{\partial h}{\partial \eta} = D\frac{1}{\eta}\frac{\partial h}{\partial \eta} + D\frac{\partial^2 h}{\partial \eta^2}. \tag{46}$$

This equation is invariant under the transformation $\eta \to -\eta$, by this we expect an even solution. Inserting the following polynomial expression into the differential equation of h

$$h(\eta) = A_0 + A_1\eta^2 + ... \tag{47}$$

– where A_0 and A_1 are constants –, we get by direct substitution

$$A_1 = A_0\frac{1}{4D}. \tag{48}$$

The general solution of (46) is

$$h(\eta) = A_1(4D + \eta^2) + A_2 e^{-\frac{\eta^2}{4D}}\text{KummerU}\left(2, 1, \frac{\eta^2}{4D}\right). \tag{49}$$

where A_1 and A_2 are constants depending on the boundary conditions of the problem.

6. CONCLUSIONS

The work considers the two dimensional radial heat diffusion equation. The equation describes the phenomena, when diffusion occur with cylindrical symmetry. There are solutions of this equation, depending on the boundary conditions of the problem. Here we considered the case for infinite horizon, when heat can diffuse in remote areas without constraint. There are more solution of this problem, where the physically relevant ones decay in time. We also found a countable set of solutions, one of them is the basic generic Gaussian solution, and there are further solutions, which can be superposed to the basic solution depending on the conditions of the problem.

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