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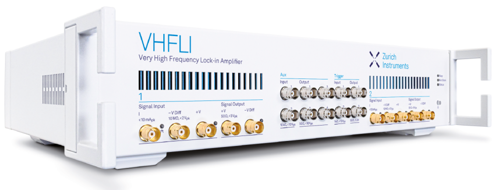

Self-similar Solutions for Diffusion Equations With Concentration-dependent Diffusion Coefficients

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Abstract. In this study, we focus on analyzing irregular diffusion equations, where the diffusion coefficients are concentration functions, leading to nonlinear behaviors. To explore these complex systems, we employ the self-similar Ansatz, a powerful method that allows us to simplify the partial differential equations into ordinary differential forms. Our research examines cases where the diffusion coefficient follows a power-law dependence on concentration. Through this analysis, we identified specific solutions for these diffusion equations. Notably, these solutions can be represented using a general analytic implicit function, providing a versatile framework for understanding the diffusion process in such systems.

INTRODUCTION

One of the most fundamental transport phenomena is diffusion, where particles move from regions of higher concentration to lower concentration. A closely related process is heat conduction, where thermal energy is transferred from hotter areas to cooler ones. Both of these phenomena are of immense importance, not only in fundamental scientific research but also in numerous engineering applications. Diffusion and heat conduction are crucial to understanding various processes, ranging from chemical reactions and material design to environmental and industrial systems. Given their significance, these processes have been the subject of extensive research, resulting in an overwhelming amount of literature on the topic, from which we mention some modern textbooks [1, 2, 3, 4].

The simplest diffusion process is the regular one, which is formulated with parabolic partial differential equation (PDE) in the well-known form of

$$\frac{\partial C(x,t)}{\partial t} = D \cdot \Delta C(x,t), \tag{1}$$

where $C(x,t)$ is the concentration, and D is the diffusion coefficient, which is a positive real constant, and Δ represents the Laplace differential operator in arbitrary dimensions in an arbitrary coordinate system. Certain boundary conditions belong to equation (1).

There are numerous solutions known for finite systems, which are often related to engineering applications [5, 6]. In the last years, with the self-similar Ansatz (which we defined a bit later), we found new types of analytic solutions on the whole axis [7]. In the following study, we broaden our analysis and investigate the nonlinear diffusion equation with the self-similar research and try to find analytic solutions.

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THEORY AND RESULTS

Our starting equation is the nonlinear diffusion equation which comes from a conservation law and reads,

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right). \tag{2}$$

If the diffusion coefficient D depends on parameter C, it will generally rely on x. So the case $D(C[x,t])$ is possible. The diffusion coefficient may depend on certain physical quantities, and it may vary depending on which phase the system is: gaseous, fluid, or solid phase. Considering the general dependence $D = \zeta(C)$

$$\frac{\partial C}{\partial t} = \zeta(C)_C \cdot \left(\frac{\partial C}{\partial x}\right)^2 + \zeta(C)\cdot\frac{\partial^2 C}{\partial x^2}. \tag{3}$$

At this point, we have to define the specific form of $\zeta(C)$; as a starting point we choose the most evident case, the power law form:

$$\zeta(C[x,t]) = a\cdot C(x,t)^n \qquad \text{where n}\varepsilon\mathbb{R} \tag{4}$$

and the constant a has the role of fixing the dimension. (For simplicity, we fix its numerical value to unity.) We apply the self-similar Ansatz

$$C(x,t) = t^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) = t^{-\alpha} f(\eta), \tag{5}$$

where $f(\eta)$ is the shape function with the reduced variable η, the two self-similar exponents α and β are responsible for the decay and spreading of the solutions if both have non-negative values. In the last decade, we generalized this kind of Ansatz to multiple spatial dimensions and applied it to the Rayleigh-Bénard convection problems [9, 10] or to the heated boundary layer equations [11].

After performing the usual algebraic manipulations we get the nonlinear ordinary differential equation (ODE) of

$$a\left(-\left[\frac{1-2\beta}{n}\right] f(\eta) - \beta\eta f(\eta)'\right) = nf^{n-1}f'^2 + f^n f'', \tag{6}$$

with the next constraint

$$-1 = -n\alpha - 2\beta, \tag{7}$$

or $\alpha = (1-2\beta)/n$. One of the self-similar and power law exponents remains free, drastically increasing the space for solutions. Different cases must be separated.

We may start with the case of $\beta = 0$, which gives us an implicit solution of

$$\int^{f(\eta)} \pm\frac{na^{2n}(2+n)}{\sqrt{-na^{2n}(2+n)(2a^{2+n}-c_1)}}da - \eta - c_2 = 0. \tag{8}$$

It turned out, after some algebra, that if all four parameters of the integral a, n, c_1, c_2 are arbitrary rational numbers, there is a definite solution which can be expressed with the ${}_2F_1()$ hypergeometric function [12]

$$\begin{aligned}
&\left(n(n+2)f(\eta)^{1+2n}\sqrt{1-\frac{2f(\eta)^{2+n}}{c1}}\right)\times\\
&{}_2F_1\left(\frac{1}{2},\frac{1+n}{2+n};1+\frac{1+n}{2+n};\frac{2f(\eta)^{2+n}}{c_1}\right)\times\\
&\left((1+n)\sqrt{-n(2+n)f(\eta)^{2n}[2f(\eta)^{2+n}-c1]}\right)^{-1} -\\
&\eta - c_2 = 0.
\end{aligned} \tag{9}$$

We have yet to find such a solution in the literature on the nonlinear diffusion equation listed above.

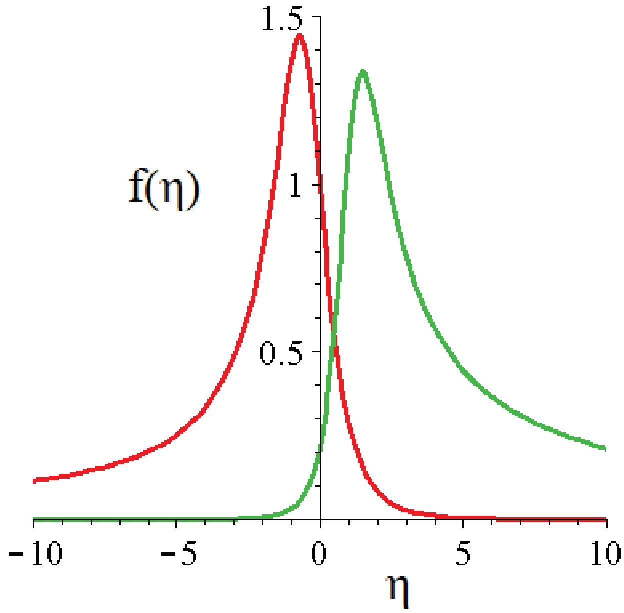

FIGURE 1. The solution of Eq. (10). The used parameter sets c_1, c_2 for the red and green lines are $(1, 2)$, and $(-2, 5)$, respectively.

An in-depth analysis of the formula is ongoing. For other parameters like, $\beta = -1$ and $n = -1$ the solution becomes much simpler

$$f(\eta) = \frac{c_1^2}{-(1 + c_1\eta) + c_1^2 c_2 e^{c_1\eta}}, \tag{10}$$

Figure 1 shows two shape functions of for different parameter sets. Note that the shape functions are zero for $\eta \to \infty$, which is physically desirable. For completeness figure (2) presents the $C(x, t)$ total solution in the form of:

$$C(x, t) = \frac{1}{t}\left(\frac{c_1^2}{-(1 + c_1\rho(x/t) + c_1^2 c_2 e^{\frac{c_1 x}{t}}}\right). \tag{11}$$

It is clear to see that the solution goes to zero at asymptotically large temporal and spatial coordinates. Further work is being done to understand all the details.

Summery and Outlook

In this study, we examined a highly nonlinear diffusion equation, where the diffusion constant—more accurately described as a concentration-dependent parameter—varies directly with the concentration of the diffusing substance. This departure from traditional diffusion models introduces significant complexity to the analysis. To address this, we employed a self-similar trial function. This powerful mathematical technique simplifies the system's behavior by reducing the partial differential equation to an ordinary one. This approach proved particularly effective for cases where the diffusion coefficient follows a power-law dependence on concentration.

Through this method, we discovered a physically relevant analytic solution that exhibits power-law decay characteristics as time tends toward infinity. These long-term behaviors suggest that the system stabilizes predictably, governed by the nonlinear interactions between the concentration and the diffusion parameter. The solution we derived not only enhances our understanding of such nonlinear diffusion processes but also provides valuable insights into their real-world applications, particularly in areas such as material science, chemical engineering, and environmental science, where diffusion with concentration dependence plays a crucial role.

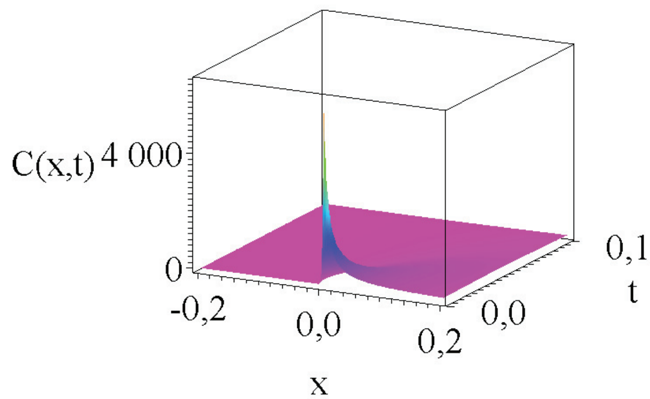

FIGURE 2. The solution of Eq. (11), the presented $C(x,t)$ function is for $\alpha = 1, \beta = 1, c_1 = 15, c_2 = 0.7$, parameter set, respectively.

Furthermore, due to the intricate relationship between the free parameters involved in the system (as illustrated in Eq. 7), there is the potential to uncover additional analytic solutions by exploring different parameter spaces. These solutions could offer new perspectives on nonlinear diffusion behaviors and expand the range of phenomena that can be modeled using this framework. As a result, ongoing work is focused on further exploring these possibilities to derive new, physically meaningful solutions that could enhance both theoretical models and practical applications of diffusion processes.

In the future - as a natural generalization - we plan to investigate reactions diffusion equations which are diffusion equations with extra source terms on the right hand side.

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