mathematics

MDPI

Article

Mathematically Exact Non-Square-Integrable Solutions in Schrödinger-Equivalent Diffusion Dynamics

László Mátyás [1] and Imre Ferenc Barna [2,*]

1 Department of Bioengineering, Faculty of Economics, Socio-Human Sciences and Engineering, Sapientia Hungarian University of Transylvania, Libertătii sq. 1, 530104 Miercurea Ciuc, Romania; matyaslaszlo@uni.sapientia.ro
2 Hungarian Research Network, Wigner Research Centre for Physics, Konkoly-Thege Miklós út 29–33, 1121 Budapest, Hungary
* Correspondence: barna.imre@wigner.hun-ren.hu

Abstract

We analyze the spherically symmetric complex diffusion and special type of the complex reaction–diffusion equations. These equations are form invariant to the free Schrödinger equations and to the Schrödinger equations with power-law space-dependent potentials. Our new type of solutions are important because we found a new realm of solutions which lie between the solutions of the classical regular diffusion equation and the usual quantum mechanical solutions of the Schrödinger equation. As the solution method, we applied the the self-similar Ansatz, which reduces the original partial differential equation (PDE) to an ordinary differential equation (ODE) which can be solved analytically. The self-similar Ansatz couples the spatial and temporal variables together instead of the usual separation which has to be used in ordinary quantum mechanics for time-independent Hamiltonian. For the complex diffusion equation—without any additional source term—the solutions are the Kummer's M and Kummer's U functions. For some parameter values we found L^2 integrability, as in the Cartesian case. We interpret that this property can be a "quantum mechanical heritage" and can be a far relation to ordinary quantum mechanics. Therefore, in this sense, our solutions might have quantum mechanical interest in the future. For the complex reaction–diffusion-type equation we derived the Whittaker M and Whittaker W functions as solutions. These solutions have no L^2 integrability at all. All derived solutions have complex quadratic arguments. These kind of analytic solutions are new and cannot be found in the existing scientific literature. Finally, the role of the complex angular momentum was investigated as well.

Keywords: Schrödinger equation; self-similar Ansatz; diffusion

MSC: 35A09; 35C06; 60J60; 81Q05

Check for updates

Academic Editor: Huaizhong Zhao

Received: 3 March 2026
Revised: 24 March 2026
Accepted: 26 March 2026
Published: 31 March 2026

1. Introduction

One of the simplest transport mechanisms is particle diffusion or heat conduction in solids. Such a process has been under investigation by physicists, mathematicians and engineers in the last two centuries. The corresponding literature is extensive and fills entire libraries; hence, we just list some recent textbooks [1–4].

It is also clear that the cornerstone of non-relativistic quantum mechanics is the Schrödinger equation (SE), which is a complex diffusion equation. The SE, therefore, describes the dispersion of the wave function of quantum particles. Such statements can be found in all basic quantum mechanic textbooks in all decades [5–8]. It is well known [9] that quantum mechanics was born exactly one hundred years ago in 1925 when Max Born first used the German phrase of "Über Quantenmechanik" in one of his papers [10]. The entire quantum theory was worked out in the later years.

In quantum mechanics, the mathematical framework of a Hilbert space is essential because it provides a natural setting in which physical states can be rigorously defined and manipulated. Specifically, quantum states are represented by wavefunctions that belong to the space L^2 integrability, meaning that their squared magnitude is finite and can be normalized to unity, allowing a consistent probabilistic interpretation. This structure ensures that inner products, which encode probabilities and expectation values, are well defined, and that operators corresponding to physical observables act in a controlled and mathematically consistent way. As a result, the connection between quantum mechanics and Hilbert spaces is not merely convenient but fundamental: the requirement of square-integrability guarantees both the physical interpretability and the mathematical stability of the theory.

The main motivation of our present study is to remember this centenary, deriving new type of results that lie between the classical diffusion equation and the quantum diffusion of a particle which is described by the Schrödinger equation. We mention that there are variational principles which lead to equations of diffusion or certain generalizations which are in connection to it [11], and its quantum counterpart may be also derived by [12].

Some of the similarities and differences between real and complex diffusion equations can be found in [13,14]. We follow this path in our next study and present self-similar solutions of the spherically symmetric SE. This analysis is organically connected to our decade-long activity in which we systematically investigated numerous diffusion or hydrodynamic equations and dissipate physical systems, and presented physically relevant disperse solutions. As an example, we just mention one of our previous works [15]. Interestingly, we note that with the self-similar Ansatz we derived hydrodynamic scaling solutions which are consistent with the Hubble parameter measurements and help to describe the rapidly expanding Universe [16]. Therefore, the scaling self-similar Ansatz has proven to be a useful method over time.

In our former study we analyzed the same problems but in Cartesian symmetry [17]. First, the free motion was investigated, and the Kummer's M and Kummer's U functions were found as results. Some of the solutions had L^2 integrability. Later, we found analytic solutions for five power-law potentials $V(x) = ax^i$, where $i = -2, -1, 0, 1, 2$, which can be expressed with some special functions, like Whittaker, Kummer or Heun functions. The a/x^2 potential has some peculiarities in ordinary quantum mechanics which are also reflected in our analysis. It became clear that there are three disjunct regimes: real diffusion with self-similar symmetry, the usual time-independent Schrödinger equation with additional potentials where temporal and spatial variables are separated, and our new complex diffusion equation where temporal and spatial variables remain coupled via the self-similar Ansatz.

A very analogous study is presented here in the following. We know from our university studies of electrodynamics or from basic quantum mechanics that the Cartesian of spherical symmetry significantly changes the results, e.g., different kind of solutions emerge. We investigate here the spherically symmetric complex diffusion equation followed by a special reaction–diffusion equation. All investigated systems have analytic solutions which are analyzed in detail.

2. Theory and Results

2.1. The Spherical Real Diffusion Equation

To have a complete overview and analysis, let us outline the self-similar solution of the real diffusion equation in curved space:

$$\frac{\partial C(r,t)}{\partial t} = D\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n \frac{\partial C(r,t)}{\partial r}\right), \tag{1}$$

where $n = 0, 1$ and 2 means Cartesian, cylindrical and spherical symmetry, respectively. For solutions, we take the Ansatz introducing the self-similar variables defined by Neumann–Sedov–Taylor [18–20] in the following form:

$$C(r,t) = t^{-\alpha} f\left(\frac{r}{t^{\beta}}\right) = t^{-\alpha} f(\eta), \tag{2}$$

where $f(\eta)$ is the shape function with existing continuous first and second derivative in respect to the reduced variable η, which is defined via $\eta = \frac{r}{t^{\beta}}$. The Ansatz have two free real parameters which are the self-similar exponents α and β. Beta is responsible for spatial spreading or hilliness of the solution and alpha is for the temporal decay or enhancement of the solutions. Physically relevant spreading and decaying solutions usually have positive α and β exponents. All these properties were exhaustively discussed in our former studies [21,22]. It is worth mentioning here that not all kinds of PDEs have self-similar symmetry automatically. This Ansatz can be generalized to PDE systems with multiple variables as well, e.g., [23]. There always exist(s) algebraic equation(s) between/among the self-similar exponents, which could lead to contradiction or contradictions which mean that self-similarity is violated and cannot be used. In such cases, additional, artificial temporal dependencies (e.g., $t^{\nu}, \;\; \nu \in \mathbb{R}\backslash 0$) can fix the problem. Whether such equations with forced time dependence have reasonable interpretation is another problem.

After some trial algebraic steps, we get the reduced ordinary differential equation (ODE) of

$$-\alpha f - \frac{\eta f'}{2} = D\left(\frac{n f'}{\eta} + f''\right), \tag{3}$$

with the usual constraints of $\alpha =$ arbitrary real and $\beta = 1/2$. The general solutions read as follows:

$$f(\eta) = e^{-\frac{\eta^2}{4D}}\left[c_1 M\left(\frac{1}{2} + \frac{n}{2} - \alpha, \frac{1}{2} + \frac{n}{2}, \frac{\eta^2}{4D}\right) + c_2 U\left(\frac{1}{2} + \frac{n}{2} - \alpha, \frac{1}{2} + \frac{n}{2}, \frac{\eta^2}{4D}\right)\right], \tag{4}$$

where $M()$ and $U()$ are the Kummer's M and Kummer's U functions [24,25]. These solutions also show certain non-Gaussian aspects of the classical dynamics [22].

The above relation holds for $n \geq 1$, and Equation (4) describes well the cylindrical and spherical symmetric solutions. Note that, unlike the Cartesian case, here, all the solutions have even property (symmetrical about the y-axis). For the spherical symmetric case we have the direct form of

$$f(\eta) = e^{-\frac{\eta^2}{4D}}\left[c_1 M\left(\frac{3}{2} - \alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right) + c_2 U\left(\frac{3}{2} - \alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right)\right]. \tag{5}$$

Without additional analysis we just give the global properties depending on the free parameter α for Kummer's M function:

- $\alpha < 0$, the solutions are divergent at large arguments;
- $\alpha = 0$, the solution is constantly 1;
- $0 < \alpha \leq 3/2$, the solutions have a local maxima and a decay to zero at large arguments;

- $3/2 < \alpha$, the solutions have oscillations proportional to the value of α and have quicker and quicker decays to zero at larger α values.

For the Kummer's U functions, the properties are a bit different:

- $\alpha < 3/2$, the solution starts at zero and has a very sharp and very high positive peak and a very quick decay to zero;
- $\alpha = 3/2$, the solution is unity;
- $3/2 < \alpha < 5/2$, the solution starts from zero has a very sharp and very high negative peak and a very quick decay to zero;
- $\alpha = 5/2$, the solutions starts from -1.5 and becomes slightly positive and has a slow decay to zero;
- $\alpha > 5/2$, the solution starts from zero then has a positive or negative very sharp peak (maximum or minimum) then an oscillatory decay. The larger the α, the larger the number of oscillations.

It is clear to see that all of the resulting functions have odd symmetry. An in-depth analysis of the self-similar solutions of the regular diffusion equation in Cartesian coordinates can be found in [22] and papers referenced therein.

2.2. The Spherical Complex Diffusion Equation

Before we derive and discuss our analytic solutions, we have to summarize other available results for the free spherical Schrödinger equation.

$$i\hbar \frac{\partial \Psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{2}{r}\frac{\partial \Psi(r,t)}{\partial r} + \frac{\partial^2 \Psi(r,t)}{\partial r^2}\right). \tag{6}$$

First, we give the form of the spherically disperse wave packet. With direct derivation and substitution, it can be proven that the solution has the form of

$$\Psi(r,t) = \left(\frac{a}{a^2 + (\hbar t/m)^2}\right)^{\frac{3}{2}} e^{-\frac{ar^2}{2(a+i\hbar t/m)}}, \tag{7}$$

and the probability density is given as

$$P(r,t) = |\Psi|^2 = \Psi^*\Psi = \left(\frac{a}{\sqrt{a^2 + (\hbar t/m)^2}}\right)^3 e^{-\frac{ar^2}{a^2+(\hbar t/m)^2}}, \tag{8}$$

where a is an arbitrary real parameter. The norm is finite and can be calculated analytically for any given parameter set. As an example, after fixing $\hbar = m = a = t = 1$,

$$\int_0^\infty P(r,t) r^2 dr = \lim_{r\to\infty} \frac{-re^{-\frac{r^2}{2}}}{2^{3/2}} + \frac{\sqrt{\pi}}{2^{5/2}} erf\left(\frac{r}{\sqrt{2}}\right) \approx 0.443. \tag{9}$$

A more general formula is also available and can be found in ([8], pp. 237–240):

$$\psi(r,t) = \left(\frac{2}{\pi w(t)^2}\right)^{3/4} \exp\left(-\frac{r^2}{w(t)^2} + i\frac{mr^2}{2\hbar t} + i\phi(t)\right), \tag{10}$$

with

$$w(t) = w_0\sqrt{1 + (i\hbar t)/(mw_0^2)}. \tag{11}$$

Defining

$$\psi(r,0) = \left(\frac{2}{\pi w_0^2}\right)^{3/4} \exp\left(-\frac{r^2}{w_0^2} + i\mathbf{k}_0 \cdot \mathbf{r}\right). \tag{12}$$

Understanding the quantum properties of matter via investigating the dynamics of wave packet motion is a popular method with an immense body of literature, so we just mention two studies [26,27]. In general, the role of time in quantum mechanics has attracted remarkable interest and topics of scientific monographs and publications, so we also give some references [28–30]. Here, we should mention the work of Kleber, who summarized the exact solutions for time-dependent phenomena in quantum mechanics [31]. Various self-similar types of solutions of different Schrödinger equations can be found in [32–35].

Changing to our method, we apply the self-similar Ansatz in the form of $\Psi(r,t) = t^{-\alpha} g\left(\frac{r}{t^\beta}\right) = t^{-\alpha} g(\omega)$ [19], and we immediately arrive to a similar ODE of

$$i\left(-\alpha g - \frac{\omega g'}{2}\right) = -D\left(\frac{2g'}{\omega} + g''\right). \tag{13}$$

At this point, we have two possibilities.

If we expect the function g to be real, then in Equation (13) the l.h.s., which is complex, may equal the r.h.s., which is real, only in the case if both equal zero:

$$-\alpha g - \frac{\omega g'}{2} = 0, \qquad \frac{2g'}{\omega} + g'' = 0. \tag{14}$$

After some rearrangement, the equations have the following differential form:

$$\frac{dg}{d\omega} = -\frac{2\alpha}{\omega}, \qquad \omega\frac{df}{d\omega} = -2f, \tag{15}$$

where in the second equation, $g' = f$ was used. Solving these two equations, one gets

$$g = g_0\left(\frac{\omega_0}{\omega}\right)^{2\alpha}, \qquad f = f_0\left(\frac{\omega_0}{\omega}\right)^2, \tag{16}$$

where g_0, f_0 and ω_0 are constants related to initial conditions $g(\omega_0) = g_0$. Because $g' = f$, integrating the second equation, one finds that

$$g = g_0\left(\frac{\omega_0}{\omega}\right)^{2\alpha}, \qquad g = -\frac{f_0\omega_0^2}{\omega}. \tag{17}$$

These two functions may be the same, if they have the same decay in ω. This condition can be fulfilled, if $\alpha = \frac{1}{2}$, The general solution in this case reads

$$\Psi(r,t) = t^{-\frac{1}{2}} \cdot g(\omega) = t^{-\frac{1}{2}} \cdot \text{Const} \cdot \frac{1}{\omega} = \text{Const} \cdot t^{-\frac{1}{2}} \cdot \frac{t^\beta}{r} = \text{Const} \cdot \frac{t^{-\frac{1}{2}+\beta}}{r}. \tag{18}$$

If the function g may be a general complex function—it is not required to be real—then the usual constraints of $\alpha =$ arbitrary rea, and $\beta = \frac{1}{2}$ hold, with diffusion constant $D = \frac{\hbar}{2m}$. The solutions of (13) are

$$g(\omega) = c_1 M\left(\alpha, \frac{3}{2}, \frac{i\omega^2}{4D}\right) + c_2 U\left(\alpha, \frac{3}{2}, \frac{i\omega^2}{4D}\right), \tag{19}$$

where $M()$ and $U()$ are still the Kummer's function [24,25]. Note that the two relevant differences to the Cartesian solutions are the lack of the extra ω dependence and the shift in the first argument of the Kummer's functions. It is useful to evaluate the first few terms of the Taylor series of the function which read as follows:

$$g(\omega) = c_1\left(1 + \frac{i\alpha\omega^2}{6D} - \frac{\alpha[\alpha+1]\omega^4}{120D^2} - \frac{i\alpha[\alpha+1][\alpha+2]\omega^6}{5040D^3} + \ldots\right) + c_2\left(\frac{2\sqrt{\pi}}{\sqrt{\frac{i}{D}}\Gamma[\alpha]} \cdot \frac{1}{\omega} - \frac{2\sqrt{\pi}}{\Gamma[-\frac{1}{2}+\alpha]} + \frac{i(-i+2\alpha)\sqrt{\pi}}{2\sqrt{\frac{i}{D}}D^2\Gamma[\alpha]} \cdot \omega - \frac{i\alpha\sqrt{\pi}}{3\Gamma[-\frac{1}{2}+\alpha]D} \cdot \omega^2 - \frac{[-1+4\alpha^2]\sqrt{\pi}}{48\sqrt{\frac{i}{D}}D^2\Gamma[\alpha]} \cdot \omega^3 + \ldots\right). \tag{20}$$

Note that the Kummer's M function has even symmetry. The Kummer's U function, however, has no even or odd symmetry for general αs. However, due to the gamma function for negative αs, the odd terms are not undefined, and for negative half-integer values the even terms are undefined.

For completeness we give the final analytic solution in the following form:

$$\Psi(r,t) = t^{-\alpha}\left(c_1 M\left[\alpha, \frac{3}{2}, \frac{ir^2}{4Dt}\right] + c_2 U\left[\alpha, \frac{3}{2}, \frac{ir^2}{4Dt}\right]\right). \tag{21}$$

To understand the properties of this solution, we have to make a regular parameter study, namely, how the solution depends on the free parameter α. Figures 1 and 2 show the real, imaginary and absolute value of the shape functions for different αs for the Kummer's M and for the Kummer's U functions. The first figure shows the solutions which are regular at the origin. The real and the imaginary parts look similar to each other and it is hard to say which α value divides the different branches of solutions. The absolute value figures, however, help us to resolve the issue unambiguously. Positive αs define shape functions with decaying and oscillatory properties. $\alpha = 0$ means the constant solution, and negative αs define divergent solutions. The second figure presents the irregular Kummer's U function which (most of them) are infinite in the origin. Positive αs define a solution which are divergent in the origin and has a strong decay to zero.

For completeness, Figure 3 presents two possible radial particle density functions $r^2|\Psi(r,t)|^2$ expressions. Note that the Kummer's M function shows some temporal wavy structure which is familiar from the "ordinary" quantum mechanic solutions. The Kummer's U functions (which are the irregular solutions) have a very quick decay in space and time with a high value in the origin. To investigate the possible far relationship to quantum mechanics, we numerically integrated the $r^2|\Psi(r,t)|^2$ quantity for numerous αs and parameters. In the former Cartesian case study [17], we found an α parameter region, where the spatial numerical integral (which is the norm of the wave function) becomes convergent for the Kummer's U function. There was no numerical convergence found for the Kummer's M functions. Now—for the spherical symmetric case—with interval doubling, we can easily prove the same property. There is no convergence for the Kummer's M function at any α parameter. For Kummer's U function, we found that for $\alpha > 0.8$ the convergence is easy to see. We cannot say at which α value lies the convergence radius.

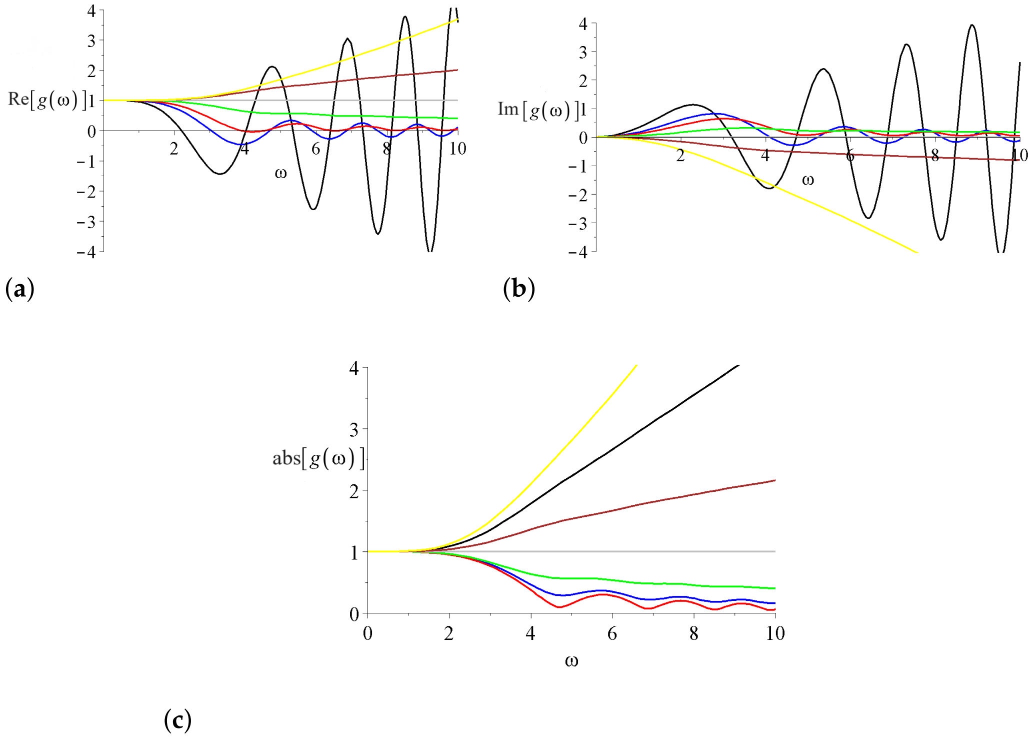


Figure 1. The (**a**–**c**) are the real, the complex, and the absolute values of the Kummer's M function in Equation (19). The black, blue, red, green, gray, brown, and yellow lines are for $\alpha = 1, 1/2, 1/4, 0, -1/4, -1/2$, and -1, respectively.

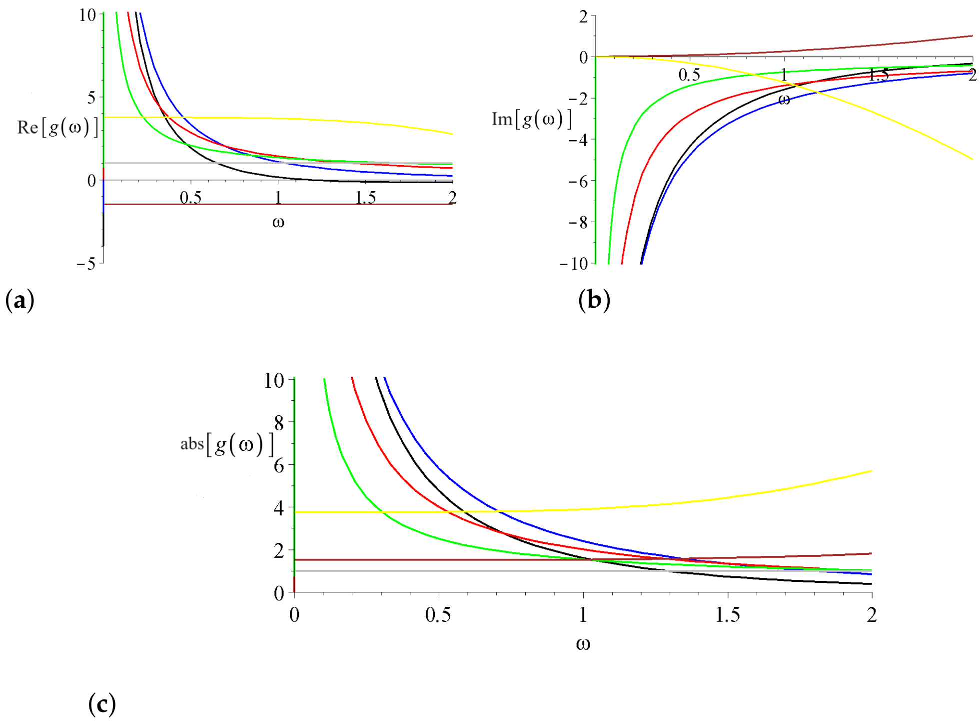
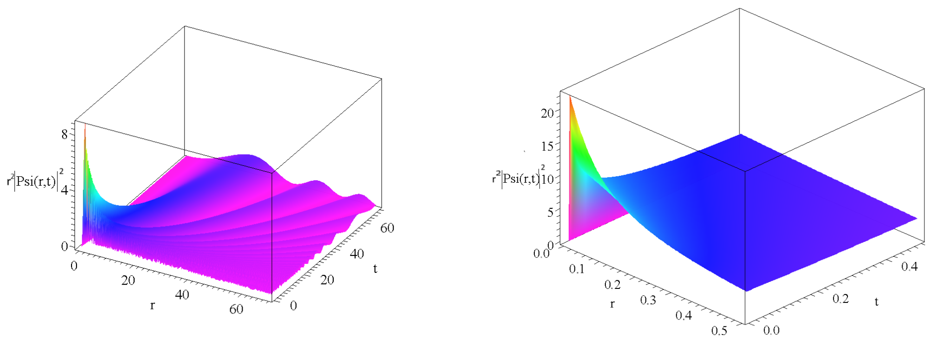


Figure 2. The (**a**–**c**) are the real, the complex, and the absolute values of the Kummer's U function in Equation (19). The black, blue, red, green, gray, brown, and yellow lines are for $\alpha = 1, 1/2, 1/4, 0, -1/4, -1/2$, and -1, respectively.



(**a**) (**b**)

Figure 3. The $r^2|\Psi(r,t)|^2$ with Equation (21) for $\alpha = 2/3$ and $\hat{D} = 1/2$. (**a**) shows the Kummer's M and (**b**) the Kummer's U function, respectively.

2.3. The Complex Spherical Reaction–Diffusion Equation

Carry forward the analysis employing the three-dimensional, spherically symmetric complex reaction–diffusion equation in its canonical Schrödinger equation form:

$$i\hbar\frac{\partial\Psi(r,\theta,\varphi)}{\partial t} = -\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2 sin(\theta)}\right)\left[sin(\theta)\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(sin(\theta)\frac{\partial\Psi}{\partial\Theta}\right) + \frac{1}{sin(\theta)}\frac{\partial^2\Psi}{\partial\varphi^2}\right] + V(r)\Psi(r,\Theta,\varphi), \tag{22}$$

where μ is the reduced mass of the two-body problem and $V(r)$ is the applied potential. To come closer to the time dependent diffusion equation, consider the $E \sim i\hbar\frac{\partial}{\partial t}$ operator substitution from now on. With the separation of the temporal variables it is possible to reduce to the pure radial coordinate, which has the final form of

$$i\hbar\frac{\partial\Phi(r,t)}{\partial t} = -\frac{\hbar^2}{2\mu}\left(\frac{2}{r}\frac{\partial\Phi(r,t)}{\partial r} + \frac{\partial^2\Phi(r,t)}{\partial r^2}\right) + V(r)\Phi(r,t) + \frac{l(l+1)\hbar^2}{2\mu r^2}\Phi(r,t), \tag{23}$$

and the technical details can be found in basic textbooks [5–8]. For completeness, we just mention that one of the authors investigated the single- and double-ionization of helium atoms with the time-dependent coupled-channel methods in a mixed Slater and regular Coulomb wave packet basis in heavy ion collisions [36] and in short intense laser fields [37] for many years. Therefore, we practically know the standard methods, and how a real quantum mechanical problem should be handled to derive observable quantities.

At this point, without completeness, we try to summarize the literature of analytic solutions for spherical potentials.

Beckers et al. [38] investigated non-relativistic quantum mechanical equations with the subgroups of the Euclidean group, where the Schrödinger or the the Pauli equations were examined with different scalar and vector potentials.

The solutions for the most common interactions like finite-depth square well, Woods–Saxon, spherical oscillator, Coulomb, Hulthén, Kratzer's molecular, Morse, Yukawa, and exponential potentials can be found in the book of Flügge [39]. For work related to certain features of solutions of biharmonic non-linear Schrödinger equations, one may consult ref. [40].

The inverse-square-root potential $V(r) = -a/r^{-\frac{1}{2}}$ was solved by Li and Dai [41] in 2016 and they derived the biconfluent Heun functions as solutions [42]. A class of exactly solvable rationally extended non-central potentials in two and three dimensions were investigated by Kumari et al. [42].

In addition to the Coulomb and the harmonic oscillator problem, the $V(x) = -a/x^2$ potential has exotic properties and shows some anomalies in quantum mechanics [43]. The problem has a remarkable body of literature; therefore, we mention just the relevant studies [44–48]. The potential has direct applications in classical celestial mechanics [49] or even in cosmology matter near the horizon of a black hole [50,51]. This potential appears in such physical problems as the Efimov effect [52], an electron near a bipolar molecule [53–55], and a neutral atom in the electric field of a thin charged wire [56,57]. It is crucial to emphasize that these studies do not mention our time-dependent self-similar solutions.

To give a later comparison to our self-similar solution, we now show the regular quantum mechanical solution of the a/r^2 potential to non-zero angular momenta. The potential does not depend on time, so the energy of the system is conserved; therefore, the next time-independent Schrödinger equation (which is an ordinary differential equation) should be solved:

$$-D\left(\frac{2}{r}\frac{\partial \Phi(r)}{\partial r} + \frac{\partial^2 \Phi(r)}{\partial r^2}\right) + \frac{a\Phi(r)}{r^2} + \frac{l(l+1)}{r^2}\Phi(r) = E\Phi(r,t), \tag{24}$$

where E is the energy of the system, a is the strength of the interaction potential (could be attractive or repulsive as well), and l is the angular momenta, now with positive integer values. The solutions are well known as the Bessel functions in the first (J) and second (Y) kind with the following form:

$$\Phi(r) = c_1\sqrt{r}J\left(\frac{\sqrt{D+4a+4l^2+4l}}{2\sqrt{D}}, \sqrt{\frac{E}{D}}r\right) + c_2\sqrt{r}Y\left(\frac{\sqrt{D+4a+4l^2+4l}}{2\sqrt{D}}, \sqrt{\frac{E}{D}}r\right), \tag{25}$$

where c_1, c_2 are the usual real integration constants.

Let us consider now Equation (23) as a complex diffusion equation and solve it with the self-similar Ansatz. In our case, the constraints for α and β exponents dictate that only the $V(r) = \frac{a}{r^2}$ static potential is available to derive a self-similar ODE in the form of

$$i\left(-\alpha h - \frac{\omega h'}{2}\right) = -D\left(\frac{2h'}{\omega} + h''\right) + \frac{a + l(l+1)b}{\omega^2}h, \tag{26}$$

where $b = \frac{\hbar^2}{2\mu}$, and we allow another free parameter a, which is the potential strength.

We have to make an important statement at this point. Our reduction mechanism makes possible any kind of time-dependent power-law type potential in the form of $V_n(r,t) = \hat{a}r^n t^{-n-2}$ where $\hat{a}, n \in \mathbb{R}$ can be reduced to an ODE, similar to Equation (26). We found analytic solutions for the $n = -2, -1, 0, 1$, and 2 exponents only. The other solutions can be expressed with the Whittaker, Kummer's and Heun functions.

We concentrate now on the $n = -2$ case exclusively. The question of the Coulomb problem—which also has an analytic solution that contains the Heun functions multiplied by an exponential and a power law functions—could be the task of a future study. Purely η-dependent potentials could be also considered, like Gaussian $e^{-\eta^2}$ or a Cauchy–Lorentzian $a/(1+\eta^2)$, and can be investigated as well.

The solution of Equation (26) reads

$$h(\omega) = \frac{1}{\eta^{3/2}}\left(c_1 e^{\frac{i\omega^2}{8D}} M_{\frac{3}{4}-\alpha, \frac{\sqrt{(4l^2+4l)b+D+4a}}{4\sqrt{D}}}\left[\frac{i\omega^2}{4D}\right] + c_2 e^{\frac{i\omega^2}{8D}} W_{\frac{3}{4}-\alpha, \frac{\sqrt{(4l^2+4l)+D+4a}}{4\sqrt{D}}}\left[\frac{i\omega^2}{4D}\right]\right), \tag{27}$$

where $M()$ and $W()$ are the Whittaker functions. For more information, please consult [24,25]. Let us analyze (27) first in detail.

Note that both the Whittaker function and the Gaussian prefactor have complex quadratic arguments. First, consider the the $l = 0$ case and study the α dependence.

Figures 4 and 5 present the real, imaginary, and absolute value of the shape function $h(\omega)$ for the Whittaker M and Whittaker W functions for various αs for zero angular momenta. It is, again, clear to see that αs of about $1/2$ have the most reasonable decaying properties. Figure 6 presents the absolute value squared of both Whittaker functions. These are, in a sense, similar to Figure 3. The Wittaker M function shows a more wavy-like structure than the Whittaker W function. We could not find convincing convergence for the numerical integral of the absolute squared Whittaker W and Whittaker M functions at any α.

For completeness, we investigate the role of the angular momenta also for the more relevant self-similar exponent of $\alpha = 1/2$. To enhance transparency and reduce the number of presented figures and curves, we just present the angular dependence of the absolute value of the Whittaker M and Whittaker W functions. We know from the physics of the Coulomb problem ($V(r) = -a/r$) that the larger the angular momentum, the larger and the more expansive the wave function is in space. Figure 7 presents these two functions for $l = 0, 1, 2, 3$ and 4. The trend is clear to see.

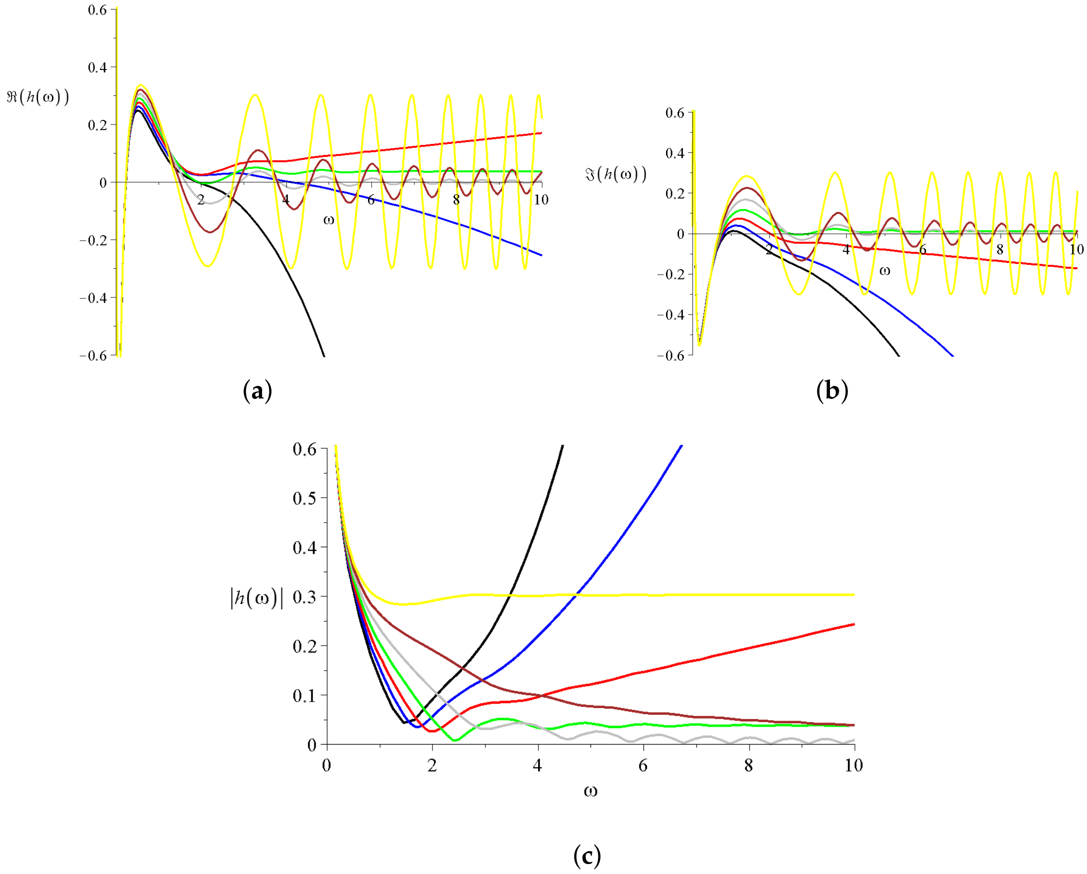


Figure 4. The (**a**–**c**) are the real, the complex, and the absolute values of the Whittaker M shape function in Equation (27) with $l = 0$. The black, blue, red, green, gray, brown, and yellow lines are for $\alpha = -3/2, -1, -1/2, 0, 1/2, 1$, and $3/2$, respectively.

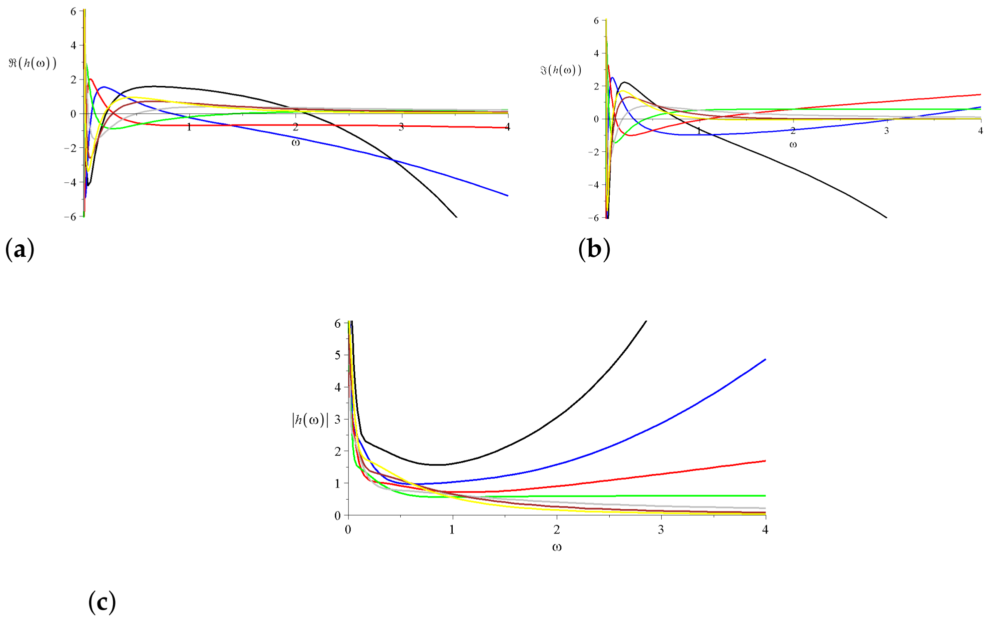


Figure 5. The (**a**–**c**) are the real, the complex, and the absolute values of the Whittaker W shape function in Equation (27) with $l = 0$. The black, blue, red, green, gray, brown, and yellow lines are for $\alpha = -3/2, -1, -1/2, 0, 1/2, 1,$ and $3/2$, respectively.

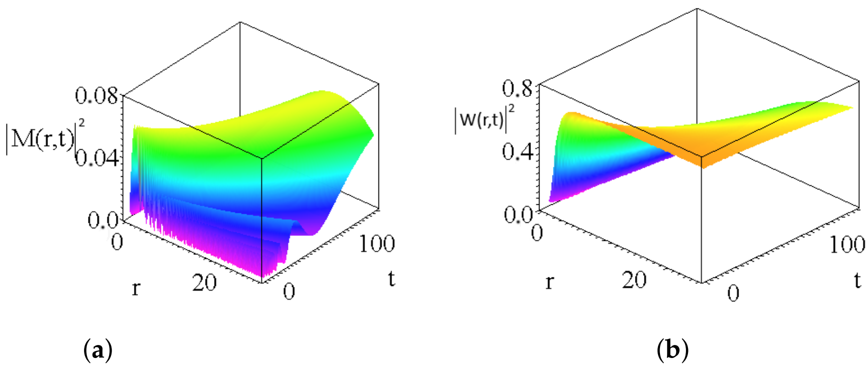


Figure 6. The absolute value square of the Whittaker M (**a**) and Whittaker W (**b**) functions of Equation (27), respectively. Other parameters are $D = 1/2$, $\alpha = 1/2, l = 0, a = -1, b = 1$.

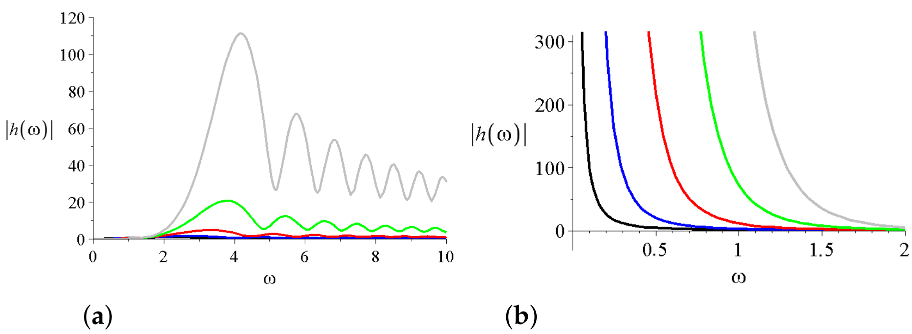


Figure 7. The absolute values of the Whittaker M (**a**) and Whittaker W functions shape functions (**b**). The black, blue, red, green , and gray lines represent $l = 0, 1, 2, 3, 4$, respectively. Other parameters are $D = 1/2$, $\alpha = 1/2$, and $a = b = 1$.

Finally, we analyze the most sophisticated case, where the reduced ODE has an additional arbitrary complex parameter. Let us call it $\hat{l} = i \cdot p + q$, where p is real number responsible for the imaginary part, and q is a second real number responsible for the real part. Now the ODE reads as follows:

$$i\left(-\alpha k - \frac{\omega k'}{2}\right) = -D\left(\frac{2k'}{\omega} + k''\right) + \frac{a + (i \cdot p + q)(i \cdot p + q + 1)b}{\omega^2}k, \tag{28}$$

The solution looks very similar, but due to the two new free parameters p, q, it becomes more elaborate:

$$k(\omega) = \frac{1}{\eta^{3/2}}\left(c_1 e^{\frac{i\omega^2}{8D}} \mathbf{M}_{\frac{3}{4}-\alpha,\theta}\left[\frac{i\omega^2}{4D}\right] + c_2 e^{\frac{i\omega^2}{8D}} \mathbf{W}_{\frac{3}{4}-\alpha,\theta}\left[\frac{i\omega^2}{4D}\right]\right), \tag{29}$$

For better transparency we use the θ abbreviation for the second parameter of the Whittaker function.

$$\theta = \frac{\sqrt{(-4p^2 + [8Iq + 4I]p + 4q^2 + 4q)b + D + 4a}}{4\sqrt{D}}. \tag{30}$$

Why are we doing this, and why could it be interesting? In quantum mechanics, these are the complex angular quantum numbers: $\hat{l} = i \cdot p + q$. The idea of complex angular momentum was introduced not because angular momentum is physically complex, but as a powerful mathematical extension of partial-wave analysis in scattering theory. Traditionally, scattering amplitudes are expanded in a sum over integer angular momentum quantum numbers, but this representation becomes unwieldy and opaque at high energies. In 1959, Tullio Regge [58] realized that by analytically continuing the angular momentum variable into the complex plane—an idea rooted in complex analysis—one could reinterpret the infinite sum as a function whose behavior is governed by singularities (poles) in that plane. These so-called Regge poles encode the physical content of the system, linking resonances and particle families to trajectories that relate spin and energy, and providing a far more unified and predictive description of high-energy scattering phenomena. To learn more about the subject, one might start with the older books, like those from Collins [59], Gribov [60], Frautschi [61], or from Omnés and Froissart [62]. The general literature of the Regge theory of the spherical inverse square potential was worked out by Mastalir in a series of publications [63–65]. He investigated a special $1/r^2$ potential which is regularized in the origin to a numerical value of $-V_0$ and at infinity to $-V_2$. The distributions of the Regge poles are given at low energies with analytic formulas.

The complex angular momenta automatically produced a pole structure on the complex plane, which helped us to understand the resonance structures and physical properties of high-energy particles.

Here, our point of view is different. We have a special reaction–diffusion equation which is form invariant to the Schrödinger equation which has a term with complex angular momenta. Our solution $\Phi(r,t)$—due to the self-similar Ansatz—has a time dependence which should not be the case in ordinary quantum mechanics when the potential is time-independent. Therefore, Equation (29) is not a quantum mechanical wavefunction, it is just the solution of a very special complex diffusion equation. We now cannot define the S matrix or the Jost function, which are essential for the investigation of the Regge poles; we can just check how our solution behaves on the complex angular momenta plane. We may hope that this solution "inherited something from quantum mechanics" and may have some very interesting feature. Therefore, if we fix all the numerical values of the physical parameters α, D, a, b and the temporal 'r' and and spatial variables 't', then we can study the role of the real and complex parts of the angular momenta. Usually, we choose $\alpha = 1/2, D = 1/2$, where the inverse square potential is attractive $a = -1$, and the angular momentum term is positive $b = +1$. Figure 8 shows how the solutions of Equation (29) depend on the real and imaginary parts of the angular momenta (notated with p and q).

https://doi.org/10.3390/math14071162

Unfortunately, we found no drastic dependence, no remarkable structure, and no poles on the complex plane. As the two independent variables grow, so too grows the Whittaker W function. The Whittaker M function behaves a bit different; it grows with enhancing q variable at $p = 0$. We could not find an extra angular momentum dependence feature in this sense.

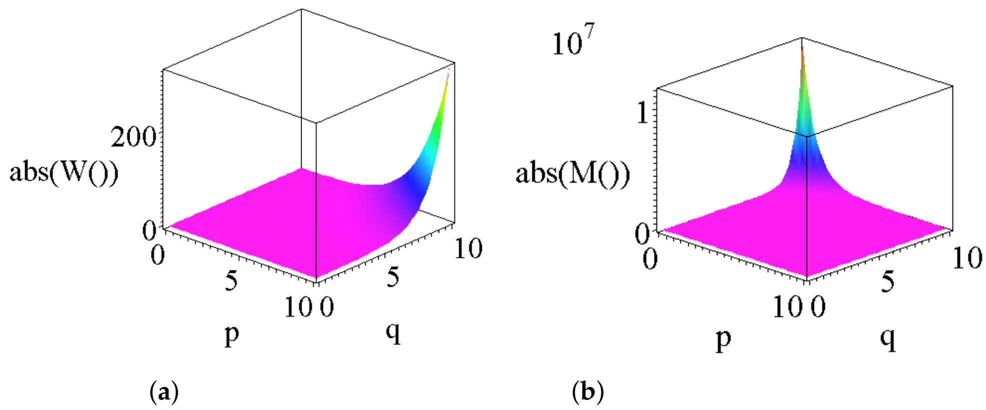


Figure 8. Graphs of the angular momentum dependence of Equation (29) with the parameter set of $\alpha = 1/2, D = 1/2, a = -1, b = 1, r = 5, t = 1$. The first subfigure (**a**) presents the absolute value of the Whittaker W and the second subfigure (**b**) for the Whittaker M function, where p is the real and q is the imaginary value of the angular momenta, respectively.

3. Summary and Outlook

First, we presented the general self-similar solutions for the regular diffusion equation as spherical coordinates, and the global properties were explained. After that, we investigated the self-similar solutions for the one-dimensional complex diffusion equation in the spherical coordinate system. This equation is form invariant to the free Schrödinger equation, but our solution method couples temporal and spatial variables; therefore, these kinds of solutions cannot be interpreted quantum mechanically. These solutions can be expressed with the Kummer's M and Kummer's U function with complex and quadratic arguments. For some α values—which is the new parameter of the solutions—even L^2 integrability can be achieved, which mimics—we may say that a kind of quantum mechanical property was "inherited".

In the rest of the study, some power-law types of potentials were added to the right hand side of the equations; the solutions remained analytic expressible with the Kummer's, Whittaker, or Heun functions. We concentrated on the inverse square potential, which is the only time-independent potential for the self-similar Ansatz. No numerical integrability was found for the absolute square of the solutions for any kind of α parameter.

In the last part of our study, we considered a complex parameter in our complex diffusion equation—which is the complex angular momenta in ordinary quantum mechanics—together with the $U(r) = \frac{a}{r^2}$ potential. The derived results remained analytic expressible with the Whittaker functions. For us, it is remarkable that even such a kind of complicated complex ODE with so many free parameters has an analytic solution. The second parameter of the the Whittaker functions became complicated and can have an even more complex value. We investigated whether the complex angular momenta can cause any kind of singularities, poles, or interesting unusual property in the solutions. In regular quantum mechanics, complex angular momenta cause poles in the S-matrix [58], which explains the resonance structure of quantum systems at high energy. Here, in this analysis, no such feature was found; the solutions were smooth and increased strictly monotonically with enhancing values of the angular momenta. Therefore, no "quantum-type of property" was inherited for this system.

Unfortunately, we cannot give physical interpretation or any other physical application to our self-similar solutions either in quantum mechanics or in complex diffusion. We think that beyond the real regular diffusion and the usual quantum diffusion (which is governed by the Schrödinger equation), we found a new realm of solutions for the complex diffusion equation. That is the main result of our presented study.

We believe that our new type of solution might prove useful in the far distant future for currently speculative theories, such as quantum consciousness, originally introduced by Penrose and Hameroff [66], as well as for other refined or extended theories discussed later in [67]. The possible connection between mind and quantum mechanics is a large, interdisciplinary , and speculative field with an exhaustive body of literature, so we just mention two summary study books [68,69].

As future straightforward investigations, we may mention the addition of variable reduction Ansatz (which describes non-classical symmetries) and this is available in [21]. As two examples, we may consider the "self-similar power series expansion" like

$$\Phi(r,t) = t^{-\alpha}\left[a \cdot f\left(\frac{r}{t^{\beta}}\right) + b \cdot f\left(\frac{r}{t^{\beta}}\right)\right]^2, \quad (31)$$

which gave us remarkable solutions with compact support to the regular diffusion equation [21]. It might happen that solutions with compact support can be derived for complex diffusion equations. As a second method, we may try the traveling profile Ansatz from Benhamidouche [70], which interpolates between the two physically relevant solutions of the disperse self-similar and the traveling wave:

$$\Phi(r,t) = t^{-\alpha} \cdot f\left(\frac{r - c \cdot t}{t^{\beta}}\right). \quad (32)$$

Additionally, the complexification of the self-similar Ansatz can be considered also, like

$$\Phi(r,t) = t^{-\alpha} \cdot f\left(\frac{ir}{t^{\beta}}\right). \quad (33)$$

All these ideas can be be studied in the future.

Author Contributions: Conceptualization, I.F.B. and L.M.; methodology, I.F.B.; software, I.F.B.; validation, L.M.; formal analysis, L.M. and I.F.B.; explanation, L.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: All the needed and relevant data are included in the text.

Acknowledgments: We would like thank the help of Sándor Varró giving us information about the properties of the wave packets.

Conflicts of Interest: The authors declare no conflicts of interest. All presented results were developed independently by the two authors.

References

1. Crank, J. *The Mathematics of Diffusion*; Clarendon Press: Oxford, UK, 1956.
2. Ghez, R. *Diffusion Phenomena*; Dover Publication Inc: New York, NY, USA, 2001.
3. Benett, T. *Transport by Advection and Diffusion: Momentum, Heat and Mass Transfer*; John Wiley & Sons: Hoboken, NJ, USA, 2013.
4. Newman, J.; Battaglia, V. *The Newman Lectures on Transport Phenomena*; Jenny Stanford Publishing: Singapore, 2021.
5. Schiff, L.I. *Quantum Mechanics*; McGraw-Hill: New York, NY, USA, 1969.
6. Sakurai, J.J. *Modern Quantum Mechanics (Revised Edition)*; Addison-Wesley: Boston, MA, USA, 1993.
7. Messiah, A. *Quantum Mechanics*; North-Holland Publishing Company: Amsterdam, The Netherlands, 1961.

8. Claude Cohen-Tannoudji, B.D.; Laloë, F. *Quantum Mechanics, Volume 1: Basic Concepts, Tools, and Applications*; Wiley-VCH: Weinheim, Germany, 2019.
9. Fedak, W.A.; Prentis, J.J. The 1925 Born and Jordan paper "On quantum mechanics". *Am. J. Phys.* **2009**, *77*, 128–139. [CrossRef]
10. Born, M. Über Quantenmechanik. *Z. Phys.* **1924**, *26*, 379–395. [CrossRef]
11. Penrose, O.; Fife, P.C. Thermodynamically consistent models of phase-field type for the kinetics of phase transitions. *Phys. D* **1990**, *43*, 44. [CrossRef]
12. Ván, P. Holographic fluids: A thermodynamic road to quatnumt physics. *Phys. Fluids* **2023**, *35*, 057105. [CrossRef]
13. Nagasawa, M. *Schrödinger Equation and Diffusion Theory*; Springer: Berlin/Heidelberg, Germany, 1993.
14. Aebi, R. *Schrödinger Diffusion Processes*; Springer: Berlin/Heidelberg, Germany, 2007.
15. Barna, I.F.; Bognár, G.; Mátyás, L.; Hriczó, K. Self-similar analysis of the time-dependent compressible and incompressible boundary layers including heat conduction. *J. Therm. Anal. Calorim.* **2022**, *147*, 13625–13632. [CrossRef]
16. Szigeti, B.E.; Szapudi, I.; Barna, I.F.; Barnaföldi, G.G. Can rotation solve the Hubble Puzzle? *Mon. Not. R. Astron. Soc.* **2025**, *538*, 3038–3041. [CrossRef]
17. Barna, I.F.; Mátyás, L. Analytic Solutions of the Complex Diffusion Equation with Aspects on Quantum Mechanics. *Int. J. Mod. Phys. A* **2025**, *40*, 2542009. [CrossRef]
18. von Neumann, J. The point source solution. In *Collected Works*; Taub, A.H., Ed.; Pergamon: New York, NY, USA, 1963; Volume 6, pp. 219–237.
19. Sedov, L.I. *Similarity and Dimensional Methods in Mechanics*; CRC Press: Boca Raton, FL, USA, 1993.
20. Taylor, G.I. The formation of a blast wave by a very intense explosion. *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1950**, *201*, 175–186.
21. Barna, I.F.; Mátyás, L. Advanced Analytic Self-Similar Solutions of Regular and Irregular Diffusion Equations. *Mathematics* **2022**, *10*, 3281. [CrossRef]
22. Mátyás, L.; Barna, I.F. Even and Odd Self-Similar Solutions of the Diffusion Equation for Infinite Horizon. *Universe* **2023**, *9*, 264. [CrossRef]
23. Barna, I.F.; Mátyás, L. Analytic solutions for the three-dimensional compressible Navier–Stokes equation. *Fluid Dyn. Res.* **2014**, *46*, 055508. [CrossRef]
24. Abramowitz, M.; Stegun, I.E. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; Dover Publisher: New York, NY, USA, 1970.
25. Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (Eds.) *NIST Handbook of Mathematical Functions*; Cambridge University Press: Cambridge, UK, 2010.
26. Garraway, B.M.; Suominen, K.A. Wave-packet dynamics: New physics and chemistry in femto-time. *Rep. Prog. Phys.* **1995**, *58*, 365. [CrossRef]
27. Briggs, J.S. Trajectories and the perception of classicalmotion in the free propagation of wave packets. *Nat. Sci.* **2022**, *2*, e20210089. [CrossRef]
28. Muga, J.G.; Sala Mayato, R.; Egusquiza, I.L. *Time in Quantum Mechanics*; Springer: Berlin/Heidelberg, Germany, 2002.
29. Penrose, R.; Isham, C.J. *Quantum Concepts in Space and Time*; Clarendon Press: Oxford, UK, 1986.
30. Briggs, J.S.; Rost, J.M. Time dependence in quantum mechanics. *Eur. Phys. J. D* **2000**, *10*, 311. [CrossRef]
31. Kleber, M. Exact solutions for time-dependent phenomena in quantum mechanics. *Phys. Rep.* **1994**, *236*, 331–393. [CrossRef]
32. Cazenave, T.; Weissler, F.B. More self-similar solutions of the nonlinear Schrödinger equation. *Nonlinear Differ. Eq. Appl. NoDEA* **1998**, *5*, 355–365. [CrossRef]
33. Fujiwara, K.; Georgiev, V.; Ozawa, T. Self-similar solutions to the derivative nonlinear Schrödinger equation. *J. Differ. Eq.* **2020**, *268*, 7940–7961. [CrossRef]
34. Pérez-García, V.M. Self-similar solutions and collective coordinate methods for Nonlinear Schrodinger Equations. *Phys. D Nonlinear Phenom.* **2003**, *191*, 211–218. [CrossRef]
35. Gutierrez, S.; Vega, L. Self-similar solutions of the localized induction approximation: Singularity formation. *Nonlinearity* **2004**, *17*, 2091. [CrossRef]
36. Barna, I.F.; Tőkési, K.; Burgdörfer, J. Single and double ionization of helium in heavy-ion impact. *J. Phys. B At. Mol. Opt. Phy.* **2005**, *38*, 1001–1013. [CrossRef]
37. Barna, I.F.; Rost, J.M. Photoionisation of helium with ultrashort XUV laser pulses. *Eur. Phys. J. D* **2003**, *27*, 287–290. [CrossRef]
38. Beckers, J.; Patera, J.; Perroud, M.; Winternitz, P. Subgroups of the Euclidean group and symmetry breaking in nonrelativistic quantum mechanics. *J. Math. Phys.* **1977**, *18*, 72–83. [CrossRef]
39. Flügge, S. *Practical Quantum Mechanics*; Springer: Berlin/Heidelberg, Germany, 1999.
40. Farkas, C.; Mezei, I.I.; Nagy, Z.T. Multiple solution for a fourth-order nonlinear eigenvalue problem with singular and sublinear potential. *Stud. Univ. Babeș-Bolyai Math.* **2023**, *68*, 139–149. [CrossRef]
41. Li, W.D.; Dai, W.S. Exact solution of inverse-square-root potential V(r) = $-\alpha$r. *Ann. Phys.* **2016**, *373*, 207–215. [CrossRef]

42. Kumari, N.; Yadav, R.K.; Khare, A.; Mandal, B.P. A class of exactly solvable rationally extended non-central potentials in two and three dimensions. *J. Math. Phys.* **2018**, *59*, 062103. [CrossRef]
43. Coon, S.A.; Holstein, B.R. Anomalies in quantum mechanics: The 1/r2 potential. *Am. J. Phys.* **2002**, *70*, 513–519. [CrossRef]
44. Guggenheim, E.A. The inverse square potential field. *Proc. Phys. Soc.* **1966**, *89*, 491. [CrossRef]
45. Gupta, K.S.; Rajeev, S.G. Renormalization in quantum mechanics. *Phys. Rev. D* **1993**, *48*, 5940–5945. [CrossRef]
46. Vasyuta, V.M.; Tkachuk, V.M. Falling of a quantum particle in an inverse square attractivepotential. *Eur. Phys. J. D* **2016**, *70*, 267. [CrossRef]
47. Guillaumín-España, E.; Núñez-Yépez, H.N.; Salas-Brito, A.L. Classical and quantum dynamics in an inverse square potential. *J. Math. Phys.* **2014**, *55*, 103509. [CrossRef]
48. Martínez-y Romero, R.P.; Núñez-Yépez, H.N.; Salas-Brito, A.L. The two dimensional motion of a particle in an inverse square potential: Classical and quantum aspects. *J. Math. Phys.* **2013**, *54*, 053509. [CrossRef]
49. Arnold, V.I. *Mathematical Aspects of Classical and Celestial Mechanics*; Springer: New York, NY, USA, 1997.
50. Chakrabarti, S.K.; Gupta, K.S.; Sen, S. Universal near-horizon conformal structure and black hole entropy. *Int. J. Mod. Phys. A* **2008**, *23*, 2547–2561. [CrossRef]
51. Camblong, H.E.; Ordóñez, C.R. Conformal tightness of holographic scaling in black hole thermodynamics. *Class. Quantum Gravity* **2013**, *30*, 175007. [CrossRef]
52. Efimov, V.N. Weakly bound states of three resonantly interacting particles. *Sov. J. Nucl. Phys* **1971**, *12*, 589.
53. Bawin, M.; Coon, S.A.; Holstein, B.R. Anions and anomalies. *Int. J. Mod. Phys. A* **2007**, *22*, 4901–4910. [CrossRef]
54. Alhaidari, A.D. Charged particle in the field of an electric quadrupole in two dimensions. *J. Phys. A Math. Theor.* **2007**, *40*, 14843. [CrossRef]
55. Giri, P.R.; Gupta, K.S.; Meljanac, S.; Samsarov, A. Electron capture and scaling anomaly in polar molecules. *Phys. Lett. A* **2008**, *372*, 2967–2970. [CrossRef]
56. Hau, L.V.; Burns, M.M.; Golovchenko, J.A. Bound states of guided matter waves: An atom and a charged wire. *Phys. Rev. A* **1992**, *45*, 6468–6478. [CrossRef]
57. Denschlag, J.; Umshaus, G.; Schmiedmayer, J. Probing a Singular Potential with Cold Atoms: A Neutral Atom and a Charged Wire. *Phys. Rev. Lett.* **1998**, *81*, 737–741. [CrossRef]
58. Regge, T. Introduction to Complex Orbital Momenta. *Il Nuovo C.* **1959**, *14*, 951. [CrossRef]
59. Collins, P. *An Introduction to Regge Theory & High Energy Physics*; Cambridge Universtity Press: Cambridge, UK, 2023.
60. Gribov, V. *The Theory of Complex Angular Momenta*; Cambridge Universtity Press: Cambridge, UK, 2003.
61. Frautschi, S.C. *Regge Poles and S-Matrix Theory*; W. A. Benjamin, Inc.: Richfield, OH, USA, 1963.
62. Omnés, R.; Froissart, M. *Mandelstam Theory and Regge Poles*; W. A. Benjamin, Inc.: Richfield, OH, USA, 1963.
63. Mastalir, R.O. Theory of Regge poles for 1/r2 potentials. I. *J. Math. Phys.* **1975**, *16*, 743–748. [CrossRef]
64. Mastalir, R.O. Theory of Regge poles for 1/r2 potentials. II. An exactly solvable example at zero energy. *J. Math. Phys.* **1975**, *16*, 749–751. [CrossRef]
65. Mastalir, R.O. Theory of Regge poles for 1/r2 potentials. III. An exact solution of Schrödinger's equation for arbitrary l and E. *J. Math. Phys.* **1975**, *16*, 752–755. [CrossRef]
66. Hameroff, S.; Penrose, R. Consciousness in the universe: A review of the 'Orch OR' theory. *Phys. Life Rev.* **2014**, *11*, 39–78. [CrossRef]
67. Derakhshani, M.; Diósi, L.; Laubenstein, M.; Piscicchia, K.; Curceanu, C. At the crossroad of the search for spontaneous radiation and the Orch OR consciousness theory. *Phys. Life Rev.* **2022**, *42*, 8–14. [CrossRef]
68. de Barros, J.A.; Montemayor, C. *Quanta and Mind: Essays on the Connection between Quantum Mechanics and Consciousness*; Springer: Berlin/Heidelberg, Germany, 2019.
69. Gao, S. *Consciousness and Quantum Mechanics*; Oxford University Press: Oxford, UK, 2023.
70. Benhamidouche, N. Exact solutions to some nonlinear PDEs, travelling profiles method. *Electron. J. Qual. Theory Differ. Eq.* **2008**, *15*, 1–7. [CrossRef]