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Analytical Examination of the Time-dependent Incompressible Boundary Layer with Heat Conduction

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Abstract. We investigate the incompressible boundary layer flow with heat conduction applying the two dimensional self-similar approximation. We found analytic solutions for the incompressible system. The parameter dependences are studied and discussed in details.

INTRODUCTION

The study of hydrodynamical equations has a crucial role in engineering and science as well. It is clear that there are numerous classification available for various flow systems. One class of fluid flows is the field of boundary layer. The development of this scientific field started with the pioneering work of Prandtl [1] who used scaling arguments that half of the terms of the Naiver-Stokes equations are negligible in boundary layer flows. In 1908, Blasius [2] gave the solutions of the steady-state incompressible two-dimensional laminar boundary layer equation forms on a semi-infinite plate which is held parallel to a constant unidirectional flow. Later, Falkner and Skan [3, 4] generalized the solutions for steady two-dimensional laminar boundary layer that forms on a wedge, i.e. flows in which the plate is not parallel to the flow. An exhaustive description of the hydrodynamics of boundary layers can be found in the classical textbook of Schlichting [5] and recent applications in engineering are discussed in [6]. The mathematical properties of the corresponding partial differential equations (PDEs) attracted much interest as well. Without completeness we mention some of the available mathematical results. Libby and Fox [7] derived some solutions using perturbation method. Ma and Hui [8] gave similarity solution to the boundary layer problems. Burde [9, 10, 11] gave numerous explicit analytic solutions in the nineties. Weidman [12] presented solutions for boundary layers with additional cross flows. Ludlow and coworkers [13] evaluated solutions with similarity methods as well. Vereshchagina [14] investigated the spatial unsteady boundary layer equations with group fibering. Polyanin in his papers [15, 16] presented numerous independent solutions derived with various methods like general variable separation.

Bognár [17] applied the steady-state boundary layer flow equations for non-Newtoninan fluids and presented self-similar results. Later it was generalized [18], and the steady-state heat conduction mechanism was included in the calculations as well.

In our former studies we investigated the Rayleigh-Bénard heat conduction problem which is a full two dimensional viscous flow coupled to a heat conduction equation. Our investigation gave a reasonable explanation of the birth of the Bénard cells [19, 20, 21]. We may say that a boundary layer equation with heat conduction is a simplified version of Rayleigh-Bénard problem. These publications [17] - [21] can be considered as precursors of the present study.

Chemical reactions in boundary layer flow are studied by Chaudhary and Merkin [22, 23]. One can find interesting studies of the boundary layer flow with nanoparticles [24], with aspects on thermophoresis [25], and bioconvection [26].

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In the following, we apply the Sedov type self-simiar Ansatz [27, 28] to the original partial differential equation (PDE) system of a boundary layer with heat conduction and reduce it to an coupled non-linear ordinary differential equation (ODE) system which can be solved with quadrature giving analytic solutions for the velocity, pressure and temperature fields. Due to our knowledge, there is no self-similar solution known and analyzed for time-dependent boundary layer equations with heat conduction.

THE THEORY OF INCOMPRESSIBLE FLOW

We start with the PDE systems of two dimensional incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial p}{\partial y} = 0, \tag{2}$$

$$\rho_\infty \frac{\partial u}{\partial t} + \rho_\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} \right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}, \tag{3}$$

$$\rho_\infty c_p \frac{\partial T}{\partial t} + \rho_\infty c_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = \kappa \frac{\partial^2 T}{\partial y^2}, \tag{4}$$

where the dynamical variables are the two fluid velocity components $u(x,y,t), v(x,y,t)$ the pressure $p(x,y,t)$ and the temperature $T(x,y,t)$. The additional physical parameters are $\rho_\infty, c_p, \mu, \kappa$ the fluid density at asymptotic distances and times, the heat capacity at fixed pressure, the dynamic viscosity and the thermal diffusivity, respectively. We apply the following self-similar Ansatz for the variables

$$\begin{aligned} u(x,y,t) &= t^{-\alpha} f(\eta), \qquad v(x,y,t) = t^{-\delta} g(\eta), \\ T(x,y,t) &= t^{-\gamma} h(\eta), \qquad p(x,y,t) = t^{-\epsilon} i(\eta), \end{aligned} \tag{5}$$

with the new variable $\eta = (x+y)/t^\beta$. All the exponents $\alpha, \beta, \gamma, \delta$ are real numbers. (Solutions with integer exponents are called self-similar solutions of the first kind, non-integer exponents generate self-similar solutions of the second kind.) The shape functions f, g, h could be any continuous functions with existing first and second continuous derivatives and will be evaluated later on. The physical and geometrical interpretation of the Ansatz were exhaustively analyzed in all our former publications [19, 20, 21] therefore we neglect it. The main points are, that $\alpha, \delta, \gamma, \epsilon$ are responsible for the rate of decay and β is for the rate of spreading of the corresponding dynamical variable for positive exponents. Negative exponents (except for some pathological cases) mean unphysical, exploding and contracting solutions. The numerical values of the exponents are the following

$$\alpha = \beta = \delta = 1/2, \qquad \epsilon = 1, \qquad \gamma = \text{arbitrary real}. \tag{6}$$

Exponents with numerical values of one half mean the regular Fourier heat conduction (or Fick's diffusion) process. Half exponent values for the velocity components and unit value exponent for the pressure decay are usual for incompressible Navier-Stokes equation [29]. The obtained system of ODE reads

$$f' + g' = 0, \tag{7}$$

$$i' = 0, \tag{8}$$

$$\rho_\infty \left(-\frac{f}{2} - \frac{f'\eta}{2} \right) + \rho_\infty (ff' + gf') = \mu f'' - i', \tag{9}$$

$$\rho_\infty c_p \left(\gamma h - \frac{h'\eta}{2} \right) + \rho_\infty c_p (fh' + gh') = \kappa h'', \tag{10}$$

where prime means derivation in respect to the variable η.

Figure 1. - 4. show the general velocity and temperature shape functions for various parameter sets. The choice of these parameters are arbitrary, however we try to create the most general and most informative figures. The functions are the modification of the error function. The crucial parameter is the ratio ρ_∞/μ, if this is larger than unity than the function tends to a sharp Gaussian.

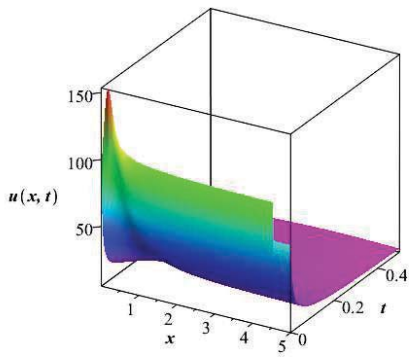

FIGURE 1. The velocity distribution function $u(x, y = 0, t) = t^{-1/2} f(\eta)$ for the following parameter set $c_1 = 1, c_2 = -2, c_3 = 1, c_4 = 1, \mu = 4.2, \rho_\infty = 1$.

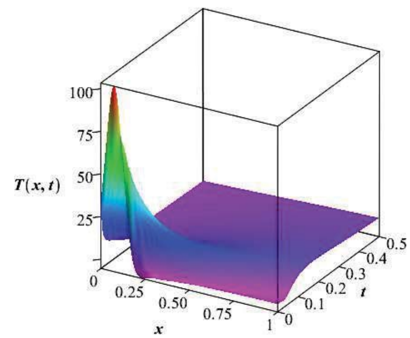

FIGURE 2. The temperature distribution function $T(x, y = 0, t) = t^{-1} h(\eta)$ for the following parameter set $c_1 = 1, c_2 = 0.4, c_3 = 1, \kappa = 0.7, c_p = 1, \rho_\infty = 1$.

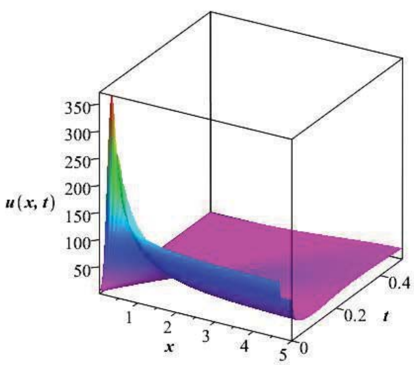

FIGURE 3. The velocity distribution function $u(x, y = 0, t) = t^{-1/2} f(\eta)$ for the following parameter set $c_1 = 3, c_2 = -2, c_3 = 1, c_4 = 1, \mu = 4.2, \rho_\infty = 1$.

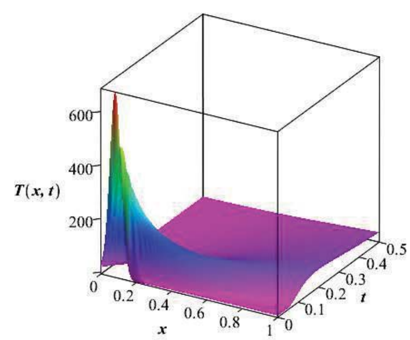

FIGURE 4. The temperature distribution function $T(x, y = 0, t) = t^{-1} h(\eta)$ for the following parameter set $c_1 = 1, c_2 = 0.4, c_3 = 1, \kappa = 0.3, c_p = 1, \rho_\infty = 1$.

SUMMARY

We analyzed the incompressible and compressible time-dependent boundary flow equations with additional heat conduction mechanism with the self-similar Ansatz. Analytic solutions were derived for the incompressible flow. The velocity and pressure fields can be expressed with the error functions (in some special cases with Gaussian functions) and the temperature with the Kummer functions. The last one has the complex mathematical structure including some oscillations. In the second part of our study, we investigated the compressible time-dependent boundary flow equations with additional heat conduction mechanism again with the self-similar Ansatz. For closing constitutive equation the ideal gas EOS was used. It is impossible to derive analytic solutions for the dynamical variables from the coupled ODE system. However, highly non-linear independent ODEs exist for each dynamical variables which can be integrated numerically. Work is in progress to apply our method to more complex flow systems like non-Newtonian fluids or non ideal gases.

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