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Self-similar analysis of a viscous heated Oberbeck–Boussinesq flow system

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Abstract

One of the simplest model to couple viscous flow to heat conduction is the Oberbeck–Boussinesq (OB) system which were also investigated by E N Lorenz. In our former studies—2015 Chaos Solitons Fractals **78** 249, 2017 Chaos Solitons Fractals **103** 336—we derived analytic solutions for the velocity, pressure and temperature fields. Additionally, we gave a possible explanation of the Rayleigh–Bénard convection cells with the help of the self-similar Ansatz. Now we generalize the OB hydrodynamical system, including a viscous source term in the heat conduction equation. Our analysis show that the viscous heating term smooths out any kind of Bénard oscillations and stabilizes the flow. All the velocity, pressure and temperature distributions are free of oscillations. These results may attract interest in micro or nanofluidics.

Keywords: Rayleigh–Bènard convection, self-similar solution, Oberbeck–Boussinesq approximation, viscous heating

1. Introduction

The investigation of coupled viscous flow equations to heat conduction has a fifteen-decade long history started with Boussinesq (1871) and Oberbeck (1879) (OB) who applied it to the normal atmosphere. At the beginning of the sixties—with the help of the stream function—Saltzman (1962) analyzed the problem with the help of finite Fourier series. At the same time Lorenz (1963) derived from the Boussinesq approximation the system which bears his name. Lorenz and Saltzman both transformed the original nonlinear partial differential equation

(PDE) system to a coupled nonlinear ordinary differential equation (ODE) system via a truncated Fourier series.

In our first study in this field (Barna and László 2015) we analyzed the original OB PDE system with the self-similar Ansatz ending up with a nonlinear ODE system, however the pressure, temperature and velocity field was evaluated in analytic forms with the help of the error functions. As main result the possible birth of the Rayleigh–Bénard (RB) convection cells was observed. In our second study (Barna *et al* 2017) we generalized the original OB hydrodynamical system, going beyond the first order Boussinesq approximation and consider a nonlinear temperature coupling. At this point more general, power law dependent fluid viscosity or heat conduction material equations were applied. The connection of the self-similar Ansatz to critical phenomena, scaling, and renormalization was addressed also.

Detailed physical description and exhausted technical details about the field of RB convection can be found in numerous books (Koschmieder 1993, Getling 1998, Kh 2009, Ching 2014, Goluskin 2016). Front propagation in RB systems—which can be investigated with the help of traveling waves—was written in details in the review study of van Saarloos (2003). Pattern formation in dynamical and non-equilibrium systems is another relevant and never-ending research field (Cross and Greenside 2009) where RB convection is one of the most investigated phenomena (Meyer-Spasche 1991). The chaotic advection phenomena can be properly modeled and described with the RB system as well (Aref *et al* 2017). Additional advection phenomena in chaotic systems can be studied in Toroczkai *et al* (1998), Boccaletti *et al* (2000). The system of equations studied in Barna and László (2015) and Barna *et al* (2017) may also contain other terms, which in the first approximation are absent because of the (initial, boundary, etc) conditions, or just neglected from the practical point of view of the problem. The Navier–Stokes equation may contain couple stresses discussed in Harfash and Meften (2018), Khan and Yousafzai (2014) or the case of a transverse seepage is analyzed in Akinaga *et al* (2016). The heat conduction equation also may contain sources or sinks, however a natural source term is the viscous heating (de Groot and Mazur 1984, Mátyás *et al* 2001, Tél *et al* 2001). Certain forms of Boussinesq description are analyzed by Ivanov and Melnikov (2015).

The thermal boundary layers can be considered as a reasonable physical simplification of our present model and was investigated by Bognár and Hriczó (2011, 2012) with self-similar and other numerical methods.

There is a considerable analytical and numerical effort to solve Boussinesq approximations or similar forms both for waves (Madsen *et al* 2006, Wazwaz 2007, 2008, Kolkovska and Dimova 2012, Shi *et al* 2012, Roeber and Cheung 2012, Wazwaz 2012, Helal *et al* 2014, Yang *et al* 2017, Kazolea and Delis 2018) and for dissipative dynamics with possible density variations (Danchin and Paicu 2009, Gastine *et al* 2015, Animasaun 2016, Lappa and Gradinscak 2018, Weiss *et al* 2018). Experiments for certain parameter values are also realized (Xi *et al* 2006, Ahlers *et al* 2012, 2014). Connections related to radiation and environment one may find in Parodi *et al* (2003).

The self-similar Ansatz and related constructions have been effectively applied in a number of hydrodynamics systems (Barna and Mátyás 2013, 2014, Yuen 2015, Chen *et al* 2017, Gugat and Ulbrich 2017, Vishwakarma *et al* 2018, Animasaun and Pop 2017). The book of Campos (Barna 2017) covers more methods related to Navier–Stokes equations. With the help of additional Fourier transformation of the analytic velocity field connections to turbulence or enstrophy could be evaluated as well. (The expression of enstrophy is not to confused with entropy. The former one is a relevant physical quantity to describe dissipation in two dimensional flows. The enstrophy can be evaluated as the integral of the square of the vorticity or with other words the integral of the square of the gradient of the velocity field.) In

the present study the original OB system in generalized in another way, with an additional viscous heating term as a source in the heat conduction equation. We perform a complete self-similar analysis of this modified OB systems and discuss the results. We show that the viscous term smooth out all the oscillatory behavior. To the best of our knowledge, there is no such study available in the literature.

Certain nonlinear dynamical systems are also able to model the viscous heating even at the level of entropy balance (Mátyás *et al* 2001). Beyond the self-similar Ansatz one can find other methods to solve hydrodynamics equations (Saad *et al* 2018).

Viscous heating plays a crucial role in the field of micro and nanofluidics (Squires and Quake 2005, Hooman and Ejlali 2010, Zhang *et al* 2010). Exhaustive description of viscous heating from that point of view can be found in the monographs of D Li (Morini 2014) and Gad-el-Hak (2006). This phenomena has relevance in other disciplines like high temperature plasma physics (Haines *et al* 2006) or magma flow in geology (Costa and Macedonio 2003). The mathematical properties of these kind of PDE equations attract some interest as well, Li (2003) investigated the global well posedness and formulated some theorems.

This paper contains a self-similar analysis of the modified OB systems where a viscous heating term is included in the heat conduction equation.

'In section 2 we present the equations of motion, which describe the system studied. Beyond this on the system of PDEs a self-similar transformation is performed. We searched for solutions of the resulting system of ODEs. The main massage of this study is that viscous heating suppress oscillations and Bénard instability if the self-similar Ansatz is applied. Section 3 presents a summary of results and outlook on possible further studies.'

2. Theory and results

Let us define our field of interest as the original OB (Oberbeck 1879, Saltzman 1962) PDS system with the additional viscous heat source term as follows

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} - eGT_1 - \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) = 0,$$

$$\frac{\partial T_1}{\partial t} + u\frac{\partial T_1}{\partial x} + w\frac{\partial T_1}{\partial z} - \kappa\left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial z^2}\right) - a\left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad (1)$$

where u, w, denote respectively the x and z velocity coordinates, T_1 is the temperature difference relative to the average ($T_1 = T - T_{av}$) and P is the scaled pressure over the density. There are four free physical parameters ν, e, G, κ the kinematic viscosity, coefficient of volume expansion, acceleration of gravitation and the coefficient of thermal diffusivity, respectively. The physical dimensions in SI units are $m^2\,s^{-1}$, C^{-1}, $m\,s^{-2}$ and $m^2\,s^{-1}$. (To avoid any misunderstanding we use the capital letter G for gravitation acceleration and g is reserved for a self-similar solution.) The quantity a regulates the strength of the viscous heating term and has a physical dimension of K s in SI units.

The first two equations are the Navier–Stokes equations, the third one is the heat conduction equation and the last one is the continuity equation. All of them contain two spatial dimensions. We apply Cartesian coordinates and Eulerian description.

We neglect the stream function reformulation of the two dimensional flow and keep the original variables investigating the original hydrodynamical system with the Ansatz of

$$u(\eta) = t^{-\alpha} f(\eta), \quad w(\eta) = t^{-\delta} g(\eta), \quad P(\eta) = t^{-\epsilon} h(\eta), \quad T_1(\eta) = t^{-\omega} l(\eta), \tag{2}$$

'where the new variable is $\eta = (x + z)/t^{\beta}$. All exponents α, β, δ, ϵ, ω are real numbers. (Solutions with integer exponents are the self-similar solutions of the first kind and sometimes can be obtained from dimensional considerations (Sedov 1993).) The f, g, h, l objects are called the shape functions of the corresponding dynamical variables. These functions should have existing first and second derivatives for the spatial coordinates and first existing derivatives for the temporal coordinate. Under certain assumptions, the PDEs describing the time propagation can be reduced to ordinary differential ones which greatly simplifies the problem. This transformation is based on the assumption that a self-similar solution exists, i.e. every physical parameter preserves its shape during the expansion. Self-similar solutions usually describe the asymptotic behavior of an unbounded or a far-field problem; the time t and the space coordinate x appear only in the combination of x/t^{β}. It means that the existence of self-similar variables implies the lack of characteristic lengths and times. These solutions are usually not unique and do not take into account the initial stage of the physical expansion process' (Barna *et al* 2017). An unwanted singularity may arises at the origin when $\eta \to 0$ which can be easily shifted with the right transformation of $\tilde{t} = t - t_0$ where t_0 is a well-chosen time interval. In the language of self-similarity it means that we may choose when we start to measure the time during the physical process.

The meaning of the exponents α, β, δ, ϵ, ω enlighten the physical picture of the Ansatz. If the one dimensional $T(x, t) = t^{-\alpha} f(x/t^{\beta})$ Ansatz is applied for the regular heat conduction equation $T_t = \kappa T_{xx}$ where subscripts mean partial derivatives the Gaussian solution can be derived with the exponents of $\alpha = \beta = 1/2$. Where α means the rate of decay and β is the spreading of the Gaussian curve.

After some algebraic manipulation of equation (1) all the critical exponents are fixed to the following values

$$\alpha = \beta = \delta = 1/2, \quad \epsilon = 1, \quad \omega = 3/2, \tag{3}$$

which are the same as in the 'original OB' system (Barna and László 2015) where no viscous heating term was considered. It is worth to mention that in the present case the fixed exponents are not enough to obtain an unambiguous ODE system, therefore an additional $1/\sqrt{t}$ time factor is needed to multiply the viscous term. Therefore, the right hand side of heat conduction equation reads $\frac{a(u_z)^2}{\sqrt{t}}$. This mean, that self-similar solutions are only available when the heating term has to have an explicit time dependence (making the entire PDE system non-autonomous) and decays at large times. This property comes from the internal logic of our dispersive self-similar Anzatz and happens sometimes. We have to denote to our former study where we tried to investigate the Maxwell–Cattaneo–Vernot telegraph heat conduction equation with the self-similar Ansatz fortunately or unfortunately we had to modify it to the Euler–Laplace–Darboux PDE equation which has self-similar solution with compact support (Barna and Kersner 2010, 2011).

The corresponding ODE now reads the following,

$$-\frac{f}{2} - \frac{f'\eta}{2} + ff' + gf' + h' - 2\nu f'' = 0,$$

$$-\frac{g}{2} - \frac{g'\eta}{2} + fg' + gg' + h' - eGl - 2\nu g'' = 0,$$

$$-\frac{3l}{2} - \frac{l'\eta}{2} + fl' + gl' - 2\kappa l'' - af'^2 = 0,$$

$$f' + g' = 0. \tag{4}$$

With straightforward algebraic manipulations, which were mentioned in our previous studies (Barna and László 2015, Barna *et al* 2017) well defined independent ODEs can be derived for the temperature, pressure and velocity shape functions. There is a hierarchy among the equations. In the original OB system the temperature is decoupled from the pressure and velocity field, and can be evaluated at first. Now, the hierarchy is changed and the velocity field became prior. The remaining ODE for the velocity shape function reads

$$8\kappa\nu f'''' - f'''[(\nu - \kappa)(4c - 2\eta)] + f''\left[6\kappa + 2c(\eta - c) - \frac{\eta^2}{2} + 16\right]$$

$$+ f'\left(\frac{\eta}{2} - c\right) + \frac{3}{2}(f - c) + eGaf'^2 = 0. \tag{5}$$

Note, that if $\nu = \kappa$ which means that the kinematic viscosity and the of thermal diffusivity are equal (which means a very peculiar system of flow) the ODE becomes incomplete. As further simplification, the integration constant which comes from the continuity equation c and can be set to zero.

Now we get an incomplete nonlinear fourth order ODE which is highly unusual. The first time derivative of the velocity is the corresponding acceleration which has physical interpretation but higher order time derivatives are meaningless (or very hard to physically interpret) in mechanical systems. (Derivation with respect to η could be considered as time-scaled coordinate or space-scaled inverse time.) This fourth order ODE is originated in the couplings mechanisms of (4).

It is trivial, that without further constraints or conditions the solutions of a fourth order ODE has a very rich mathematical structure. The original OB system describes a fluid flow in a bounded channel, therefore a mixed initial and boundary problem has to be addressed. This condition makes the problem very similar to the Prandtl boundary layer problem, which has enormous literature. Without completeness we mention the basic literature only (Schlichting and Gersten 2017). The investigation of a non-Newtonian 2D laminar boundary-layer with power-law viscosity with the self-similar Ansatz leads to a nonlinear fifth order ODE (Benlahsen *et al* 2008).

So we are interested in solutions of equation (5) where the velocities and the velocity gradients are fixed at the two boundary points. This means that the next choice is straight forward e.g. $f(0) = b_1$, $f(\eta_1) = d_1$, $f'(0) = b_2$, $f'(\eta_1) = d_2$. As a natural choice we fix the velocities to a non-zero fix value at left and zero value at the right boundary. It is also clear that a height-order nonline ODE (which is even non-autonomous, now depends even on η^2) cannot be analyzed with a full mathematical rigor, therefore we just perform *a* 'use your common sense' analysis and try to explore parameter sets where the evaluated solution behaves physically reasonable. For additionally allowed simplification we fix the value of e, G to unity and investigate the role of the viscosity ν, heat conduction κ and the strength of the viscous heating a only. Figure 1 shows the shape function of the x component of the velocity
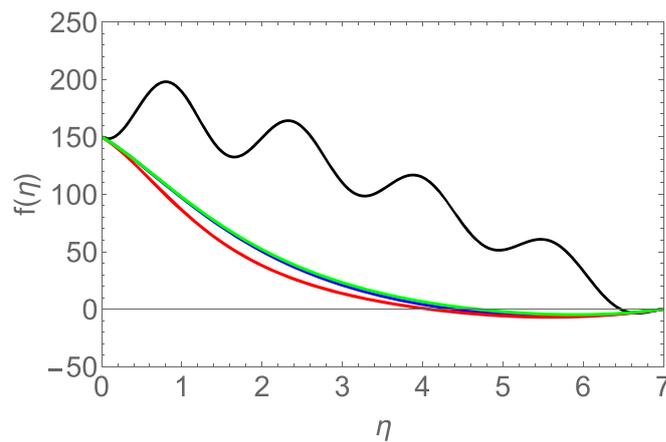
Figure 1. The shape functions of the velocity component u as the function of η for various physical parameter sets. The velocity is maximal at one boundary and zero at the other. The acceleration is negative on the maximal velocity side and positive on the other. The constants G and e are set to unity and $a = 0.5$ The black curve is for a parameter set of $\kappa = 4.1$, $\nu = 3.4$, $c = 33$. Notice the heavy oscillations. The blue, red and green curves represent results for $c = 0$ with $\kappa = \nu = 3.1$, $\kappa = 0.8 < \nu = 4.1$, and $\kappa = 5.2 > \nu = 1.8$, respectively.

between the above mentioned two boundaries for various parameter sets. We performed numerous calculations where all three parameters (ν, κ, a) lie in the closed numerical range of [0.1, 7].

We applied some build-in integration routine of Mathematica 12. with global adaptive strategy. No additional singularity handling was applied. This algorithm uses a data structure called a 'pile' to keep the set of regions partially sorted, with the largest error region being at the top of the pile. In the main loop of the algorithm the largest error region is bisected in the dimension that is estimated to be responsible for most of its error. In the presented cases the requested accuracy of precision was reached within 100 iterations.

Our important experience show, that the larger the viscosity constant of the viscous heating a the smaller the velocity in the chosen domain which meets our physical expectation. For fixed viscous heating component a the larger the values of ν, κ the larger the velocity function in the investigated domain.The most interesting feature is the role of c which is the free integration constant from the continuity equation. Usually this value is set to zero, however for non numerical zero value the velocity shape function becomes to oscillate. The higher the c value the larger the amplitudes of the oscillations. Our explanation is the following: higher c value means higher mass flow rate, which means denser fluid, and denser fluids might have larger variations in the density (even for incompressible fluids) which is a kind of external noise. So we think, that the numerical value of the free integration constant can be interpreted as the level of noise. Only this parameter causes oscillations in the velocity field, otherwise the finite values of κ, ν, a smoothen the velocity field. Smooth velocity fields prevent the formation of RB convection cells. The main message from this study at this point is that the viscous heating term (with the finite value of a) prevents any kind of instability in this model. Figure 1 presents four different velocity fields for different parameter sets. Large c values causes spurious oscillations. We investigate the role of the ratios of κ and ν if $\kappa = \nu$ the third derivative of the velocity vanishes which simplifies the ODE. The other two cases ($\kappa < \nu$) and ($\kappa > \nu$) make no difference in the final numerical results.
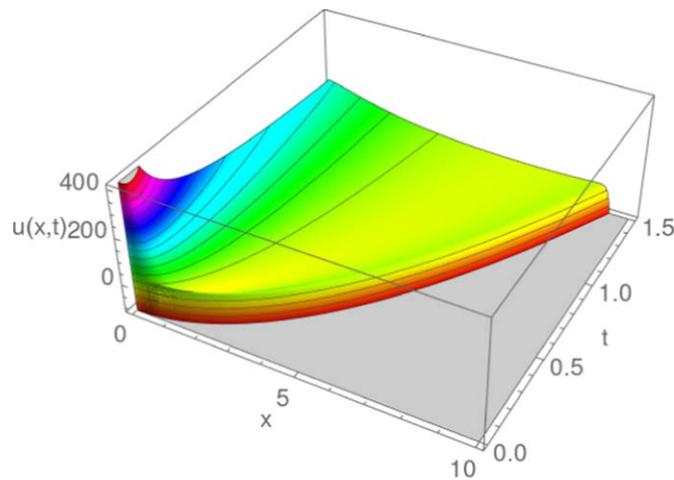
Figure 2. The velocity distribution functions $u(x, z = 0, t)$. The parameters are $c = 0$, $a = 0.5$, $\kappa = 0.8$ and $\nu = 4.1$.

For all cases presented in figure 1 the function f tends to a finite value, when $\eta \to 0$, i.e. $t \to \infty$ for finite space coordinates. Consequently the velocity function has the form

$$u \simeq \frac{\text{const.}}{t^{1/2}}, \tag{6}$$

for sufficiently large times.

Figure 2 presents the velocity function $u(x, z = 0, t)$ with $c = 0$. The distribution function is a more or less flat surface with a singularity in the origin which can be shifted with the above mentioned $\tilde{t} = t - t_0$ transformation.

The second independent ODE in the hierarchy is for the shape function of the temperature field,

$$-2\kappa l'' + l'\left(c - \frac{\eta}{2}\right) - \frac{3l}{2} - af'^2 = 0. \tag{7}$$

Note, the direct dependence on the velocity shape field derivatives. It is worth to mention that this equation is similar to equation (8) in Barna and László (2015), just modified with two extra terms of the viscous heating—proportional to a—and with a term which comes from the non-zero c integral constant of the continuity equation.

Figure 3 shows the shape function of the temperature between the same boundaries as in figure 1 with $c = 0$. The function has a clear flat minimum in the investigated interval close to zero. This means that the temperature may reach the average for a while. After sufficient long time with fixed space coordinates, when η tends to zero, the value of $l(\eta)$ approaches a fixed value, consequently based on Oberbeck (1879).

$$T_1 \simeq \frac{\text{const.}}{t^{3/2}}, \tag{8}$$

which shows the long time decay of the temperature T_1. Two different kind of boundary conditions were investigated the $l(0) = 250$, $l(7) = 200$ and the $l(0) = 150$, $l(7) = 5$ case, respectively. Is is clear that all four presented solutions are smooth. The three solutions represented with green, black and red colors belong to different κ and a values modeling different flow systems. (The numerical values of κ and a are given in the figure caption.)
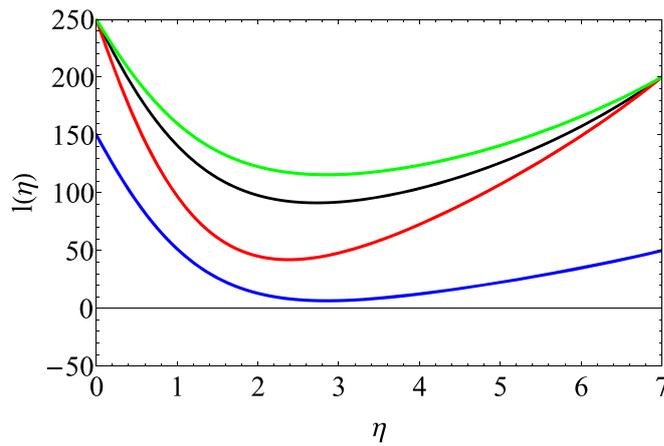
Figure 3. Shapes function of the temperature equation (7) as the function of η. Two different boundary conditions are considered which are mentioned in the text. Green, black and red curves belong to the first boundary conditions and parameter sets of $(\kappa = 4.1, a = 0.5)$, $(\kappa = 4.1, a = 1.5)$, $(\kappa = 0.8, a = 1.5)$ and the blue curve belongs to the second boundary conditions with the parameter set of $(\kappa = 4.1, a = 1.5)$, respectively. The additional parameters are $c = 0$ and $\nu = 3.4$.
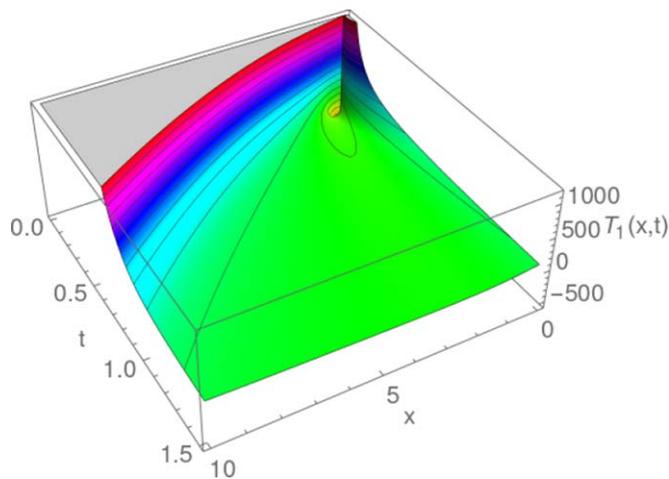
Figure 4. The temperature distribution function $T_1(x, z = 0, t)$. The parameters are the same as at figure 2.

Note, that the function is quick-decaying and missing any kind of oscillations or additional structure. Figure 4 presents the temperature distribution function to the x, t plane ($z = 0$). Note, that the function has a sharp decay missing any kind of oscillations or additional structure again.

The final ODE is for the shape function of the pressure field

$$h' = \frac{eGl}{2} + \frac{c}{4}. \tag{9}$$
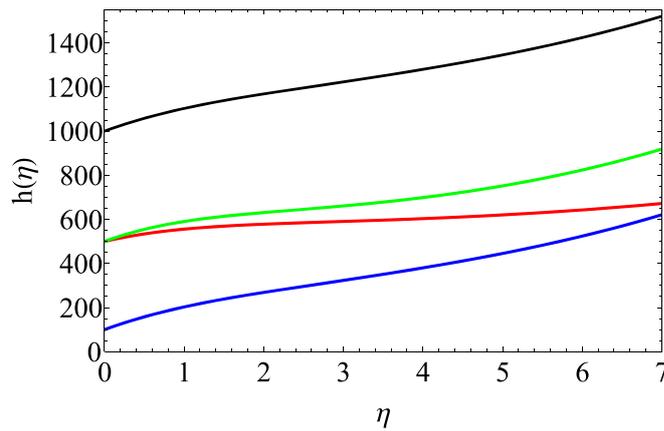
Figure 5. Various shape function of the pressure field equation (9) as the function of η. Three initial conditions are considered which are clear to see and mentioned in the text as well. The black, green, red and blue curves are calculations for ($\kappa = 4.1$, $a = 0.5$), ($\kappa = 4.1$, $a = 0.5$), ($\kappa = 0.8$, $a = 1.5$), ($\kappa = 4.1$, $a = 1.5$), respectively. The additional parameters are $c = 0$ and $\nu = 3.4$.
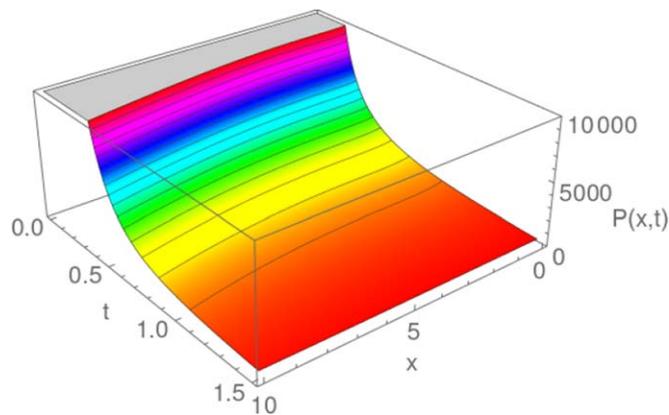
Figure 6. The pressure distribution function $P(x, y = 0, t)$. The parameters are same as at figure 2.

Figure 5 shows different shape functions of the pressure field in between two boundaries. Three different kind of initial conditions are considered $h(0) = 1000$, 500 and 100. All solutions remain stable. The numerical values of the additional applied physical parameters κ, a, ν are given in the figure caption.

The behavior of the pressure is relatively regular. For $\eta \to 0$ the function $h(\eta)$ tends to a fixed value as one can see on figure 5. This means that for sufficiently long times

$$P \simeq \frac{\text{const.}}{t}. \tag{10}$$

The decay of the pressure distribution function $P(x, z = 0, t)$ is represented on figure 6, and it has a certain monotony without oscillations.

During our present analysis of viscous heating we were speculating about additional physically relevant heating mechanisms. It is worth to mention, that we tried to find self-similar analytic solutions for radiative heating where an $a \cdot T(x, z, t)^4$ term is added to the heat conduction equation according to the well-known Stefan–Boltzmann law. The analysis of the exponents clearly showed, that an additional $t^{5/12}$ time-dependent factor is required to fulfill all necessary conditions to obtain an ODE system. Therefore we think that such a term would be non-physical therefore we skip further investigation. To analyze the original OB (Barna and László 2015) the modified OB system (Barna *et al* 2017) or even the present system with the traveling-wave Ansatz could be an additional interesting project.

We find possible that a rotation around the z axis perpendicular to the x–z plane could be an reasonable generalization as well. However, at first the effect of the rotation in the viscous fluid equations (without heat conduction) should be investigated and understood. Similar studies are already under the way.

3. Summary and outlook

We gave a physically reasonable generalization of the classical OB equation As a new feature we added an additional source term to the heat conduction equation, which is proportional to the square of the velocity gradient and called viscous heating. Instead of the usual Galerkin method which applies truncated Fourier series we took the two-dimensional generalization of the self-similar Ansatz and found a coupled nonlinear ODE system which can be solved with quadrature. The main lesson what we learned from this study is that viscous heating suppress oscillations and Bénard instability. In the following we plan to study time-dependent solutions of boundary layers which strong heat conduction.

Acknowledgments

References

Ahlers G, Bodenschatz E and He X 2014 *J. Fluid Mech.* **758** 436
Ahlers G, He X, Funfschilling D and Bodenschatz D 2012 *New J. Phys.* **14** 103012
Akinaga T, Itano T and Generalis S 2016 *Chaos Solitons Fractals* **91** 533
Animasaun I L 2016 *Ain Shams Eng. J.* **7** 755
Animasaun I L and Pop I 2017 *Alexandria Eng. J.* **56** 647
Aref H *et al* 2017 *Rev. Mod. Phys.* **89** 025007
Barna I F 2017 Self-similar analysis of various Navier–Stokes equations in two or three dimensions *Handbook on Navier–Stokes Equations, Theory and Applied Analysis* ed D Campos (New York: Nova Publishers)
Barna I F and Kersner R 2010 *J. Phys. A: Math. Theor.* **43** 375210
Barna I F and Kersner R 2011 *Adv. Stud. Theor. Phys.* **5** 193
Barna I F and László M 2015 *Chaos Solitons Fractals* **78** 249
Barna I F and Mátyás L 2013 *Miskolc Math. Notes* **14** 485
Barna I F and Mátyás L 2014 *Fluid Dyn. Res.* **46** 055508
Barna I F, Pocsai M A, Lökös S and Mátyás L 2017 *Chaos Solitons Fractals* **103** 336
Benlahsen M, Guedda M and Kersner R 2008 *Math. Comput. Modelling* **47** 1063
Boccaletti S, Grebogi C, Lai Y C, Mancini H and Maza D 2000 *Phys. Rep.* **329** 103

Bognár G and Hriczó K 2011 *Acta Polytech. Hung.* **8** 131

Bognár G and Hriczó K 2012 *Recent Advances in Fluid Mechanics, Heat and Mass Transfer* (Harvard, MA: WSEAS Press) p 198

Boussinesq M J 1871 *C. R. Acad. Sci.* **72** 755

Chen Y, Fan E and Yuen M 2017 *Appl. Math. Lett.* **67** 46

Ching E S C 2014 *Statistics and Scaling in Turbulent Rayleigh–Bènard Convection* (Berlin: Springer)

Costa A and Macedonio G 2003 *Nonlinear Process. Geophys.* **10** 545

Cross M and Greenside H 2009 *Pattern Formation and Dynamics in Nonequilibrium Systems* (Cambridge: Cambridge University Press)

Danchin R and Paicu M 2009 *Commun. Math. Phys.* **290** 1

de Groot S R and Mazur P 1984 *Non-Equilibrium Thermodynamics* (New York: Dover)

Gad-el-Hak H 2006 *The Microelectromechanical Systems (MEMS) Handbook* (London: Taylor and Francis)

Gastine T, Wicht J and Aurnou J M 2015 *J. Fluid Mech.* **778** 721

Getling A V 1998 *Rayleigh–Bènard Convection: Structures and Dynamics* (Singapore: World Scientific)

Goluskin D 2016 *Internally Heated Convection and Rayleig–Bènard Convection* (Berlin: Springer)

Gugat M and Ulbrich S 2017 *J. Math. Anal. Appl.* **454** 439

Haines M G, LePell P D, Coverdale C A, Jones B, Deeney C and Apruzese J P 2006 *Phys. Rev. Lett.* **96** 075003

Harfash A J and Meften G A 2018 *Chaos Solitons Fractals* **107** 18

Helal M A, Seadawy A R and Zekry M H 2014 *Appl. Math. Comput.* **232** 1094

Hooman K and Ejlali A 2010 *Int. Commun. Heat Mass Trasfer* **37** 34

Ivanov S E and Melnikov V G 2015 *Int. J. Math. Anal.* **9** 2659

Kazolea M and Delis A I 2018 *Eur. J. Mech.* B **72** 432

Kh R 2009 *Convection in Fluids: A Rational Analysis and Asymptotic Modelling* (Berlin: Springer)

Khan W and Yousafzai F 2014 *Adv. Trends Math.* **1** 27

Kolkovska N and Dimova M 2012 *Cent. Eur. J. Math.* **10** 1159

Koschmieder E L 1993 *Bènard Cells and Taylor Vortices* (Cambridge)

Lappa M and Gradinscak T 2018 *Int. J. Heat Mass Transfer* **121** 412

Li J 2018 arXiv:1801.09395v2 [math.AP]

Lorenz E N 1963 *J. Atmos. Sci.* **20** 130

Madsen P A, Furman D R and Wang B 2006 *Coast. Eng.* **53** 487

Mátyás L, Tél T and Vollmer J 2001 *Phys. Rev.* E **64** 056106

Meyer-Spasche R 1991 *Pattern Formation in Viscous Flow* (Berlin: Springer)

Morini G L 2014 Viscous heating *Encyklopedia of Microfluidics and Nanofluidics* ed D Li (Berlin: Springer)

Oberbeck A 1879 *Ann. Phys. Chem. Neue Feralolge* **7** 271

Parodi A, Emanuel K A and Provenzale A 2003 *New J. Phys.* **5** 106

Roeber V and Cheung K F 2012 *Coast. Eng.* **70** 1

Saad K M, Atangana A and Baleanu D 2018 *Chaos* **28** 063109

Saltzman B 1962 *J. Atmos. Sci.* **19** 329

Schlichting H and Gersten K 2017 *Boundary-Layer Theory* (Berlin: Springer)

Sedov L 1993 *Similarity and Dimensional Methods in Mechanics* (Boca Raton, FL: CRC Press)

Shi F, Kirbi J T, Harris J C, Geiman J D and Grilli S T 2012 *Ocean Modell.* **43** 36

Squires T M and Quake S R 2005 *Rev. Mod. Phys.* **77** 977

Tél T, Vollmer J and Mátyás L 2001 *Europhys. Lett.* **53** 458

Toroczkai Z, Károlyi Gy, Péntek Á, Tél T and Grebogi C 1998 *Phys. Rev. Lett.* **80** 500

van Saarloos W 2003 *Phys. Rep.* **386** 29

Vishwakarma J P, Nath G and Srivastava R K 2018 *Ain Shams Eng. J.* **9** 1717

Wazwaz A-M 2007 *J. Comput. Appl. Math.* **207** 18

Wazwaz A-M 2008 *Commun. Nonlinear Sci. Numer. Simul.* **13** 889

Wazwaz A-M 2012 *Ocean Eng.* **53** 1

Weiss S, He X, Ahlers G, Bodenschatz E and Shishkina O 2018 *J. Fluid Mech.* **851** 374

Xi H-D, Zhou Q and Xia K-Q 2006 *Phys. Rev.* E **73** 056312

Yang X-J, Machado T and Baleanu D 2017 *Fractals* **25** 1740006

Yuen M 2015 *Commun. Nonlinear Sci. Numer. Simul.* **20** 634

Zhang T, Jia L, Yiang L and Jaluria Y 2010 *Int. Commun. Heat Mass Trasfer* **53** 49272