

Asymptotic evolution of bulk-viscous, spherically symmetric spacetimesBalázs Endre Szigeti^{*}*HUN-REN Wigner Research Centre for Physics, P.O. Box 49, H-1525 Budapest, Hungary;
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The scale-free nature of gravitational interaction in both Newtonian gravity and the general theory of relativity gives rise to the concept of self-similarity, where solutions are scale invariant. As a result of this property, the governing partial differential equations are greatly simplified and can be transformed into ordinary ones. These solutions function as attractors, characterizing the asymptotic dynamics of more general solutions. There exist situations in which self-similarity is only partially realized, giving rise to kinematic self-similar solutions. Our study provides a systematic classification of kinematic self-similar solutions corresponding to the most general spherically symmetric spacetime in the presence of bulk viscous flows.

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In general relativity, self-similarity can be realized in two different ways. On one hand, it could appear as a property of spacetime and, on the other hand, as a property of matter fields [1]. Such self-similar behavior often reflects the asymptotic evolution of spacetime, offering insight into its long-term dynamics. From the mathematical cosmology aspect, the main interest has so far been focused on self-similar solutions with a perfect fluid stress-energy tensor. In this context, extending the analysis beyond the idealized perfect fluid case naturally leads to considering more general matter sources. Among these, viscous fluids provide an important generalization, as dissipative processes can play a significant role in the dynamics of cosmological models.

The description of viscosity and irreversible processes in general relativity has a long history. C. H. Eckart proposed the first model in 1940 [2]. Later, W. A. Lindblom and L. Hiscock [3] demonstrated that Eckart's proposal yields theories that are unstable and noncausal. Despite recognizing these problems since 1985, Eckart's first-order theory is widely used in cosmology. In our analysis, we have used the Landau-Lifschitz-Eckart flow, in spite of the existence of a second-order theory—proposed by Muller, Israel, and Stewart [4]—which addresses the issues arising in Eckart's

formulation [5]. K. Misner proposed the first widely recognized viscous cosmological model in Ref. [6]. He introduced viscosity in his cosmological model from a particle physics standpoint.

The great interest in viscous cosmology models emerges from one of the most fundamental concepts in hydrodynamics, that the ideal description is only an approximation of real-world systems. Moreover, as it was articulated by Maartens: “The conventional theory of the evolution of the universe includes several dissipative processes, as it must if the current large value of the entropy per baryon is to be accounted for. ... However, it is important to develop a robust model of dissipative cosmological processes in general, so that one can analyze the overall dynamics of dissipation without getting lost in the details of particular complex processes” [7]. In light of this argument, it is clear why many new viscous models have gained prominence in the scientific discourse of recent years [8,9].

Self-similarity implies that the spatial distribution of the characteristics of motion remains identical to itself at all times throughout the process. This is a well-known concept in classical physics; it was introduced by Gottfried Guderley in 1942 and used extensively in various fields of physics [10]. Also, self-similar *ansatz* represent solutions to degenerate problems in which all dimensional parameters entering the initial and boundary conditions vanish or become infinite [11]. Since then, self-similarity has become a standard tool in modern theoretical and applied physics, providing valuable insight into scale-invariant structures in systems governed by nonlinear dynamics [12,13].

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Self-similarity holds great importance in Newtonian gravitational theory, making it a useful tool for describing various systems. However, its generalization to general relativity (GR) is not quite straightforward due to its covariant nature. It was first defined by Cahill and Taub [14] and has been extensively studied for various spacetime symmetries [15,16]. It has a wide range of applications in mathematical cosmology [17,18], the gravitational collapse of black holes [19], and the study of naked singularities [20].

In Sec. II, we present the basic concepts of self-similarity in general relativity and show its application to general, nonstatic spherically symmetric spacetime. We establish the relevant equation system to find all kinematic self-similar solutions in the presence of a specific viscous fluid—using the generalization of the framework proposed by Cahill and Taub [14]. In Sec. III, we present the relevant ordinary differential equation systems arising from the self-similar *ansatz*, derive their analytic solutions, and discuss their classification in the presence of bulk viscosity terms. The basic notation and conventions used in this paper can be found in Appendix A.

II. KINEMATIC SELF-SIMILARITY OF GENERAL SPHERICAL SYMMETRIC SPACETIME

Originally, self-similarity in general relativity can be defined via the existence of a special type of Killing vector field called homothetic vector fields (HVF) [14,21]. The defining relation for HVFs is

$$\mathcal{L}_\xi g_{\mu\nu} = 2\delta g_{\mu\nu}, \quad (1)$$

where δ is a constant, the $g_{\mu\nu}$ is the metric tensor, and \mathcal{L}_ξ is the Lie-differentiation along ξ . In general, it can be proved that δ can be set to unity by rescaling the ξ vector field. Note that if δ equals zero, the homothetic vector fields become the well-known Killing vector fields. If Eq. (1) holds for a ξ homothetic vector field, then the Riemann-tensor and hence the Ricci-tensor and therefore the Einstein-tensor meet the

$$\mathcal{L}_\xi R^\mu{}_{\nu\kappa\lambda} = 0 \Rightarrow \mathcal{L}_\xi R_{\mu\nu} = 0 \Rightarrow \mathcal{L}_\xi G_{\mu\nu} = 0, \quad (2)$$

identity, respectively [22]. Contrarily, if ξ satisfies the so-called collineation equations (2), it does not mean that it is a homothetic vector field. Nonvacuum systems are described via

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \quad (3)$$

Einstein field equations (EFEs) are satisfied if the perfect fluid energy-momentum tensor must meet the collinearity condition of

$$\mathcal{L}_\xi T_{\mu\nu} = 0. \quad (4)$$

The $T_{\mu\nu}$ can be given by the standard form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \quad (5)$$

where ρ is the energy density, p denotes the pressure, and the u_μ is the four-velocity. The original article by Cahill and Taub showed that only the linear barotropic equation of state ($p = w_0\rho$) is compatible with this homothety condition for perfect fluid [14]. Comprehensive analysis of the homothetic vector field on perfect fluid was made by Eardly [17]. In his work, Eardly identified “physical” self-similarity with the “geometric” homothetic condition. However, if the source is not a perfect fluid, spacetime symmetries need not be inherited by the matter. Hence, a homothety remains a purely geometric property rather than a manifestation of self-similarity [23,24]. Exact self-similar solutions have been obtained across a variety of spacetime symmetries: plane-symmetric spacetimes [25,26], hyperbolically symmetric spacetimes [27], Weyl spacetimes [28], and spherically symmetric spacetimes [29,30], highlighting the broad applicability of self-similarity in relativistic contexts. Such models provide insight into the critical phenomena observed at the threshold of black holes [19,31] as well as primordial black hole formation [32,33].

A. Kinematic self-similar solutions

Homothetic solutions are a powerful tool for studying the solution of the Einstein Field Equations. Still, the requirement of such symmetry severely restricts the class of equations of state that allow for consistent solutions. Kinematic self-similar solutions have been introduced and used in the context of relativistic hydrodynamics, and their generalization into general relativity is an analogue of the incomplete self-similarity used in Newtonian gravitational theory [34,35]. There exists a natural generalization (proposed by Carter and Henriksen in 1989 and 1991) of homothety called kinematic self-similarity, which

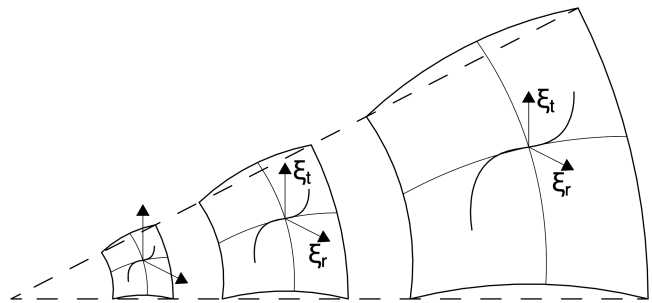


FIG. 1. Schematic view of the kinematic self-similar vector fields with local Killing vector fields, where ξ_t and ξ_r represent the local basis.

is defined by the existence of a kinematic self-similar vector field (KSSVF), as seen in Fig. 1. These kinds of vector fields must satisfy the following equations:

$$\mathcal{L}_\xi h_{\mu\nu} = 2\delta h_{\mu\nu}, \quad (6)$$

$$\mathcal{L}_\xi u_\mu = \alpha u_\mu, \quad (7)$$

where $h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$ acts as a tangential projector onto Σ_t three-dimensional spacelike hypersurface ($t(x_\mu) = \text{const.}$). The α and δ are dimensionless constants called similarity indices. These indices characterize self-similar transformations. The parameter α represents the relative proportionality factor that governs the dilation rates of the spatial length scale and the increase in the time scale. Taking into account the possible cases, the solutions are

0th kind: If both similarity indices are equal to zero, the solution is of the zeroth kind (Killing vectors).

1st kind: If the α/δ ratio is equal to unity, it is named the first kind of self-similarity [36].

2nd kind: If $\delta \neq 0$ and $\alpha \neq 0$, then it is called second-order self-similarity.

∞^{th} kind: Lastly, the case where $\delta = 0$, $\alpha \neq 0$ is an infinite kind of solution.

Several authors have explored kinematic self-similar perfect-fluid solutions [20,30,37]. We aim to investigate how these solutions are modified when viscous terms are taken into account.

B. Viscous model

In our analysis, we assumed that the spacetime is spherically symmetric but nonstatic, and then the line element can be written in the following form:

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Psi} dr^2 + R^2[d\theta^2 + \Sigma(k, \theta)^2 d\phi^2], \quad (8)$$

where

$$\Sigma(k, \theta) = \begin{cases} \sin(\theta), & \text{if } k = 1, \\ \theta, & \text{if } k = 0, \\ \sinh(\theta), & \text{if } k = -1, \end{cases} \quad (9)$$

and $\Phi = \Phi(t, r)$, $\Psi = \Psi(t, r)$, and $R = R(t, r)$ are arbitrary functions of time and radial coordinates. We use the exponential form of these functions to ensure positivity and avoid singularities. Furthermore, the exponential form often allows a more direct interpretation of these physical quantities. The function $\Phi(t, r)$ determines the time dilation effects and the gravitational potential, and $\Psi(t, r)$ accounts for the spatial curvature in the radial direction. The exponential form of the $R(t, r)$ function is deliberately avoided to maintain its direct physical interpretability and to facilitate a more meaningful comparison with the Friedmann-Robertson-Walker-Lemaître (FRWL)

model [38,39]. The energy-momentum tensor describes the viscous model in Landau-Lifschitz-Eckart flow [2,40],

$$T_{\mu\nu} = (\rho + p + \Pi)u_\mu u_\nu + (p + \Pi)g_{\mu\nu} + 2\eta\sigma_{\mu\nu}. \quad (10)$$

Introducing P , which can be considered as effective pressure modified by the bulk viscosity (Π),

$$P := p + \Pi = p - \zeta(\rho, \theta)\theta, \quad (11)$$

where $\zeta(\rho, \theta)$ is the bulk viscosity coefficient, and $\theta = u^\mu_{;\mu}$ is the expansion scalar. The shear viscosity parameter η quantifies the dissipative stress produced by anisotropic or shape-changing deformations of the fluid that occur at constant volume. We have used the linear barotropic equation of state, $p = w_0\rho$, which is compatible with the kinematic self-similarity conditions [41]. We assume that the w_0 parameter is $-1 \leq w_0 \leq 1$. One can define the $w(t, r)$ effective equation of state (EoS) parameter between the density and the effective pressure $P = w(t, r)\rho$ as

$$w(t, r) = w_0 - \frac{\zeta(\rho, \theta)}{\rho} e^{-\Phi} \left(\dot{\Psi} + 2\frac{\dot{R}}{R} \right). \quad (12)$$

The effective pressure $P(\xi)$ must satisfy $\mathcal{L}_\xi P = -2\alpha P$ based on the argument made in Appendix C. It can be seen from the definition of Eq. (11) that both terms in the sum are required to satisfy the Lie condition. Thus, $\mathcal{L}_\xi \Pi = -2\alpha \Pi$. Since θ can be expressed as the covariant derivative of u_μ and $\mathcal{L}_\xi u_\mu = \alpha u_\mu$, the $\zeta(\rho, \theta)$ should scale as $\mathcal{L}_\xi \zeta(\rho, \theta) = \alpha \zeta(\rho, \theta)$. Solutions both exhibit proper scaling and covariant nature and depend only on one of the dynamical variables, $\zeta(\rho) \propto \rho^{1/2}$ and $\zeta(\theta) \propto \theta$. Choosing $\zeta(\rho) \propto \rho^{1/2}$ has been widely employed both in cosmology, where it enables analytic treatments of anisotropic viscous models, and in neutron-star physics, where similar fractional density dependences naturally arise in weak-interaction-mediated damping of stellar oscillations [42,43]. The second approach is to let ζ depend on the expansion scalar θ , which highlights the intrinsic thermodynamic role of viscosity: the comoving expansion of the Universe itself generates bulk viscous effects [44,45], then the shear viscosity tensor goes as

$$\sigma_{\mu\nu} = \frac{1}{2} (h_{\mu\alpha} \nabla^\alpha u_\nu + h_{\nu\alpha} \nabla^\alpha u_\mu) - \frac{1}{3} h_{\mu\nu} \theta. \quad (13)$$

The decomposition of the four-velocity and its covariant derivative can be found in Appendix B, and the Lie derivatives of the stress-energy-momentum tensor can be found in Appendix C. We adopt a comoving frame, where the four-velocity takes the form of

$$u_\mu = (e^{-\Phi}, 0, 0, 0). \quad (14)$$

Thus, one can obtain the following expression for the expansion scalar:

$$\theta = e^{-\Phi} \left(\dot{\Psi} + 2 \frac{\dot{R}}{R} \right) \quad (15)$$

and the nonzero components of the shear viscosity tensor

$$\sigma_r^r = e^{-\Phi} \frac{2}{3} \left(\dot{\Psi} - \frac{\dot{R}}{R} \right), \quad (16)$$

$$\sigma_\phi^\phi = \sigma_\theta^\theta = -e^{-\Phi} \frac{1}{3} \left(\dot{\Psi} - \frac{\dot{R}}{R} \right). \quad (17)$$

Utilizing the derived expressions, the shear viscosity scalar can be defined as follows:

$$\sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} = e^{-2\Phi} \frac{1}{3} \left(\dot{\Psi} - \frac{\dot{R}}{R} \right)^2. \quad (18)$$

Consequently, the Einstein field equation expressed in terms of Misner-Sharp-Hernandez (MSH) mass and the Bianchi identity have the form

$$-P' = (\rho + P)\Phi' - (2\eta\sigma_r^r)' - 6\eta\sigma_r^r \left(\frac{R'}{R} - \frac{\Phi'}{3} \right), \quad (19)$$

$$-\dot{\rho}e^{-\Phi} = (\rho + P)\theta - 4\eta\sigma^2, \quad (20)$$

$$m' = 4\pi\rho R' R^2, \quad (21)$$

$$\dot{m} = -4\pi(P - 2\eta\sigma_r^r)\dot{R}R^2, \quad (22)$$

$$\dot{R}' = \Phi'\dot{R} + \dot{\Psi}R', \quad (23)$$

where the $m = m(t, r)$ is the MSH mass defined via the equations of [46,47],

$$g_{rr} = \left[1 + (e^{-\Phi}\dot{R})^2 - \frac{2m}{R} \right]^{-1} R^2. \quad (24)$$

The relevant components of the Ricci tensor and Ricci scalar are described in Appendix D. Also, the remaining (mixed EFEs) Eqs. (25)–(27) act as supplementary equations for the analysis

$$\begin{aligned} \kappa\rho &= e^{-2\Phi} R^{-2} [\dot{R}(\dot{R} + 2R\dot{\Psi})] + R^{-2} \\ &+ e^{-2\Psi} R^{-2} (-2RR'\Psi' + R'^2 + 2RR''), \end{aligned} \quad (25)$$

$$\begin{aligned} \kappa(P - 2\eta\sigma_r^r) &= e^{-2\Psi} R^{-2} [2RR'\Phi' + R'^2 \\ &- e^{2(\Psi-\Phi)} (e^{2\Phi} - 2R\dot{R}\dot{\Phi} + \dot{R}^2 + 2R\dot{R}\ddot{R})], \end{aligned} \quad (26)$$

$$\begin{aligned} \kappa(P - 2\eta\sigma_\theta^\theta) &= e^{-2(\Psi+\Phi)} R^{-1} [e^{2\Phi} ((R' + R\Phi')(\Phi' - \Psi') \\ &+ R'' + R\Phi'') + e^{2\Psi} ((\dot{R} + R\dot{\Psi})(\dot{\Phi} - \dot{\Psi}) \\ &- \ddot{R} - R\ddot{\Psi})]. \end{aligned} \quad (27)$$

One can assume that the particle number is conserved, which is defined as

$$\nabla_\mu N^\mu = \nabla_\mu (nu^\mu) = 0 \Rightarrow e^{-\Phi}\dot{n} + n\theta = 0, \quad (28)$$

where $n = n(t, r)$ and $N^\mu = N^\mu(t, r)$ are the particle number density and particle current, respectively. As a consequence of the definition of the $h_{\mu\nu}$ projection tensor and the condition in Eq. (6), spatial lengths scale with weight δ since the particle number density, defined as the number per spatial volume unit, must scale with δ . Hence,

$$\mathcal{L}_\xi N_\mu = (\mathcal{L}_\xi n)u_\mu + n\mathcal{L}_\xi u_\mu = (\alpha - \delta)N_\mu, \quad (29)$$

and therefore, N^μ satisfies the homothety condition.

III. RESULTS

Spherical-symmetric, time-dependent scenarios for bulk viscosity are presented in this section for various cases. Foremost, we incorporate bulk viscosity, examining its impact on dynamic evolution and its asymptotic behavior. It is known that for a general spherically symmetric spacetime, the kinematic self-similar vector field is given by the following formula:

$$\xi(t, r) = h_1(t, r) \frac{\partial}{\partial t} + h_2(t, r) \frac{\partial}{\partial r}, \quad (30)$$

where $h_1(t, r)$ and $h_2(t, r)$ are functions of the time and spatial coordinates. One solution is called *orthogonal* to the fluid flow if $h_1(t, r) = 0$, and it is *parallel* if $h_2(t, r) = 0$; otherwise, it is called *tilted*.

A. Tilted case

We begin by considering the most general scenario, where the KSSVFs are tilted and therefore not aligned either parallel or orthogonal. If we exclude the infinite kind of self-similarity, then the δ similarity parameter can be set to unity. Then, the $h_1(r, t)$ and $h_2(r, t)$ functions can be written in terms of α and β constants [48],

$$\xi(r, t) = (\alpha t + \beta) \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}. \quad (31)$$

Hence, if the α is unity, one can set the β to zero, and the kinematic self-similar vector field is equivalent to the first kind of self-similarity. In the case of the type of zeroth self-similarity, the α vanishes, and then the β is set to unity. For when α is neither zero nor unity, the β again can be fixed to zero, and the self-similarity becomes the second kind. It is summarized in Table I.

The unknown metric functions we are looking for can be expressed in terms of the similarity variable based on the dimensional analysis

TABLE I. Classifications of self-similarity types and their corresponding similarity variable [36].

Self-similarity type	Similarity variable	Similarity constants
0 th kind	$\xi = re^{-t}$	$\alpha = 0, \beta = 1$
1 st kind	$\xi = r/t$	$\alpha = 1, \beta = 0$
2 nd kind	$\xi = r(\alpha t)^{-1/\alpha}$	$\alpha \neq 0, 1, \beta = 1$
∞^{th} kind	$\xi = r/t$	

$$R(t, r) = r\mathcal{R}(\xi), \quad \Phi(t, r) = \Phi(\xi), \quad \Psi(t, r) = \Psi(\xi), \quad (32)$$

and consequently, the metric

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Psi}dr^2 + r^2\mathcal{R}^2d\Omega^2. \quad (33)$$

If $\alpha \neq 0$, the particle number density takes the canonical self-similar form of

$$n(t, r) = t^{-\delta/\alpha}\tilde{N}(\xi). \quad (34)$$

Also, if the $\alpha = 0$ and the self-similarity variable became $\xi = re^{-t}$, the previous expression varies as

$$n(t, r) = e^{-\delta t}\tilde{N}(\xi). \quad (35)$$

If the $\delta = 0$ and $\alpha = 1$, the kinematic self-similar vector becomes an infinite kind, and the vector field is

$$\xi(r, t) = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \quad (36)$$

with a self-similarity variable of $\xi = r/t$, and the metric functions are

$$R(t, r) = \mathcal{R}(\xi), \quad \Phi(t, r) = \Phi(\xi), \quad e^{\Psi(t, r)} = e^{\Psi(\xi)}/r, \quad (37)$$

which are obtained by solving the equations of Eqs. (6) and (7) for a general spherically symmetric spacetime, if the KSSVF is Eq. (36) analyzed by Sintès *et al.* [49]. For the infinite kind of self-similarity, the expression for particle number density in Eq. (34) simplifies as $n(t, r) = \tilde{N}(\xi)$.

1. Self-similarity of the second kind

It follows from Einstein's field equation that a set of ordinary differential equations is constructed for the energy density, the effective pressure, and the Misner-Shapiro mass,

$$\kappa\rho = \frac{1}{r^2} \left[\rho_0(\xi) + \frac{r^2}{t^2} \rho_2(\xi) \right], \quad (38)$$

$$\kappa P = \frac{1}{r^2} \left[P_0(\xi) + \frac{r^2}{t^2} P_2(\xi) \right], \quad (39)$$

$$2m = r \left[m_0(\xi) + \frac{r^2}{t^2} m_2(\xi) \right]. \quad (40)$$

This separation is a direct consequence of the kinematic self-similar symmetry, which permits the replacement of spatial and temporal derivatives by derivatives with respect to the similarity variable $\xi = r(\alpha t)^{-1/\alpha}$. As a result, the EFEs and matter field equations naturally split into components proportional to $\mathcal{O}[(r/t)^0]$ and $\mathcal{O}[(r/t)^2]$, which must be satisfied independently [14]. The EFEs and the conservation equations in Eqs. (19)–(23) transforms accordingly,

$$(\rho_0 + P_0)\Phi' = 2P_0 - P_0', \quad (41)$$

$$(\rho_2 + P_2)\Phi' = -P_2', \quad (42)$$

$$-\rho_0'\mathcal{R} = (\rho_0 + P_0)(\Psi'\mathcal{R} + 2\mathcal{R}'), \quad (43)$$

$$-(2\alpha\rho_2 + \rho_2')\mathcal{R} = (\rho_2 + P_2)(\Psi'\mathcal{R} + 2\mathcal{R}'), \quad (44)$$

$$m_0 + m_0' = \rho_0\mathcal{R}^2(\mathcal{R} + \mathcal{R}'), \quad (45)$$

$$3m_2 + m_2' = \rho_2\mathcal{R}^2(\mathcal{R} + \mathcal{R}'), \quad (46)$$

$$m_0' = -P_0\mathcal{R}^2\mathcal{R}', \quad (47)$$

$$2\alpha m_2 + m_2' = -P_2\mathcal{R}^2\mathcal{R}', \quad (48)$$

$$\mathcal{R}'' + \mathcal{R}' = \mathcal{R}'\Phi' + (\mathcal{R} + \mathcal{R}')\Psi', \quad (49)$$

and for the Misner-Sharp-Hernandez mass,

$$m_0 = \mathcal{R}[1 - e^{-2\Psi}(\mathcal{R} + \mathcal{R}')^2], \quad (50)$$

$$\alpha^2 m_2 = \mathcal{R}'\mathcal{R}^2 e^{-2\Phi}. \quad (51)$$

The auxiliary equations are

$$\mathcal{R}'^2 + 2\Psi'\mathcal{R}\mathcal{R}' = \alpha^2\rho_2\mathcal{R}^2 e^{2\Phi}, \quad (52)$$

$$\begin{aligned} \mathcal{R}(2\mathcal{R}'' + 4\mathcal{R}') &= 2\Psi'(\mathcal{R}^2 + \mathcal{R}\mathcal{R}') \\ &\quad - (\mathcal{R}'^2 + \mathcal{R}^2) + e^{2\Psi}(1 - \rho_0\mathcal{R}^2), \end{aligned} \quad (53)$$

$$-2\mathcal{R}(\mathcal{R}'' + (\alpha - \Phi')\mathcal{R}') = \alpha^2 P_2 \mathcal{R}^2 e^{2\Phi} + \mathcal{R}'^2, \quad (54)$$

$$(\mathcal{R}' + \mathcal{R})(\mathcal{R}' + \mathcal{R} + 2\Phi'\mathcal{R}) = e^{2\Psi}(1 + P_0\mathcal{R}^2), \quad (55)$$

where the (') denotes the derivative concerning the logarithm of the similarity variable $\ln\xi$. Similarly, an equation for the particle number density can be obtained if one substitutes the expression from Eqs. (32) and (34) into Eq. (28), which will result in

$$\frac{\tilde{N}'}{\tilde{N}} = -\frac{1}{\alpha} \left[2 \frac{\mathcal{R}'}{\mathcal{R}} + \Psi' + \delta \right]. \quad (56)$$

Integration with respect to $\ln \xi$, will lead to

$$\tilde{N} = C_N \xi^{-\delta/\alpha} \mathcal{R}^{-2/\alpha} e^{-\Psi/\alpha}. \quad (57)$$

The detailed derivation of the obtained solution can be found in Appendix E. In the first scenario, where ρ_0 vanishes, we proved that the $\zeta(\rho) \propto \rho^{1/2}$ bulk viscosity condition is not compatible with the self-similarity condition. However, in the case of $\zeta(\theta) \propto \theta$, the nonconstant metric functions in Eq. (33) are

$$\mathcal{R}(\xi) = \mathcal{R}_0 [3(1 - \tilde{U}_0) \xi^{C_U} + (1 - \tilde{U}_0)^{C_U}]^{1/3}, \quad (58)$$

$$\begin{aligned} \Psi(\xi) = & \ln(\mathcal{R}_0 \xi^{C_U/3} (3 - 2\alpha + (1 - \tilde{U}_0)^{C_U} \xi^{-C_U}), \\ & \times (3(1 - \tilde{U}_0) + (1 - \tilde{U}_0)^{C_U} \xi^{-C_U})^{-2/3}). \end{aligned} \quad (59)$$

This solution aligns with the self-similar dust models classified by Carr and Coley [29], particularly those belonging to the particular version of the Kantowski-Sachs model [50] with an appropriate set of constants. Since $\xi = r(\alpha t)^{-1/\alpha}$ and $R(t, r) = r\mathcal{R}(\xi)$, the obtained solution reduces to $\mathcal{R}(\xi) \propto \xi^{-1}$ when $C_U = -3$. In this case, the area radius behaves as $R(t, r) \propto t^{-C_U/\alpha}$, implying that the two-sphere radius does not depend on the spatial coordinates, as in the Kantowski-Sachs metric. Under this exact condition, the metric function $\Psi(\xi)$ becomes a function of t only, while $\Phi(\xi)$ remains constant. Notice that this power-law behavior holds only asymptotically in the long-time limit ($\xi \rightarrow 0$), where the power-law term becomes dominant in the metric functions. This case excludes the first-kind kinematic self-similar solution since $C_U = (3\tilde{U}_0 - 2\alpha)(1 - \tilde{U}_0)$ cannot be satisfied for any finite value of \tilde{U}_0 . In the next scenario, which is the nontrivial version of the $\rho_2 = 0$, the line element becomes

$$\begin{aligned} ds^2 = & -r^{2c_\Phi} (\alpha t)^{-\frac{2c_\Phi}{\alpha}} dt^2 + r^{-\lambda_{0,r}} (\alpha t)^{\frac{\lambda_{0,r}}{\alpha}} dr^2 \\ & + \mathcal{R}_0^2 r^{2+2\lambda_{0,r}} (\alpha t)^{-\frac{2\lambda_{0,r}}{\alpha}} d\Omega^2. \end{aligned} \quad (60)$$

By setting $c_\Phi = 0$ which implies, $\alpha = 3/2$ and consequently $\lambda_{0,r} = -1$, the metric reduces to the standard FRWL. Asymptotically FRWL solutions have been discussed in the context of PBH formation by Carr and Hawking [51] and subsequently Bicknell and Henriksen [32,52]. In the case of the so-called trivial solution, the only nonconstant term is

$$e^{2\Phi} = \xi^{\frac{4w_0}{(1+w_0)}}, \quad (61)$$

and the metric follows as

$$ds^2 = -\xi^{\frac{4w_0}{(1+w_0)}} dt^2 + \Psi_0^2 dr^2 + \mathcal{R}_0^2 r^2 d\Omega^2, \quad (62)$$

which is called a static singular solution due to its singularity at $t = 0$.

2. Self-similarity of the first kind (homothetic solution)

As in the previous case, the EFEs and the conservation equations in Eqs. (19)–(23) can be expressed in the same form as those obtained for self-similarity of the second kind. One can easily notice that this differential equation system is too restrictive. However, the homothetic solution can still be obtained as a limiting case ($\alpha = 1$) of the second-type self-similar solution.

3. Self-similarity of the zeroth kind

In the case of self-similarity of the zeroth kind, the Einstein equations imply similar constraints as it was derived in Eqs. (38)–(40),

$$\kappa\rho = \frac{1}{r^2} [\rho_0(\xi) + r^2\rho_2(\xi)], \quad (63)$$

$$\kappa p = \frac{1}{r^2} [P_0(\xi) + r^2P_2(\xi)], \quad (64)$$

$$2m = r[m_0(\xi) + r^2m_2(\xi)], \quad (65)$$

and the $\xi = re^{-t}$ similarity variable defined in Table I. The obtained equations are mostly similar to the homothetic case. In particular, Eqs. (41), (42), (45), (46) (with $\alpha = 1$), and (49), as well as the equation for the Misner-Sharp-Hernandez mass, remain equivalent. The differences arise in the remaining equations, which are as follows:

$$-\rho'_0 \mathcal{R} = (\rho_0 + P_0)(\Psi' \mathcal{R} + 2\mathcal{R}'), \quad (66)$$

$$-\rho'_2 \mathcal{R} = (\rho_2 + P_2)(\Psi' \mathcal{R} + 2\mathcal{R}'), \quad (67)$$

$$-m'_0 = P_0 \mathcal{R}^2 \mathcal{R}', \quad (68)$$

$$-m'_2 = P_2 \mathcal{R}^2 \mathcal{R}'. \quad (69)$$

The auxiliary equations of Eqs. (52)–(55) and particle number density equations of Eq. (34) are reduced to the form, where α becomes unity. Thus, we have received the solution

$$\tilde{N}(\xi) = C_{\tilde{N}} \xi^{-\delta} \mathcal{R}^{-2} e^{-\Psi}. \quad (70)$$

For the $\rho_0 = 0$ case, we obtained the following metric:

$$ds^2 = -\Phi_0^2 dt^2 + 4\mathcal{R}_0^2 e^{-2t} dr^2 + \mathcal{R}_0^2 r^2 e^{-2t} d\Omega^2. \quad (71)$$

The detailed derivation can be seen in Appendix E 2. However, the solutions reduce to the perfect fluid solution

since the θ and ρ_2 become constant. If ρ_2 vanishes, we will arrive at the following metric:

$$ds^2 = e^{2c_\Phi} (re^{-t})^{2w_0/(1+w_0)} dt^2 + \frac{c_{\mathcal{R}}^2}{c_{\mathcal{R}} - c_m} dr^2 + r^2 c_{\mathcal{R}}^2 d\Omega^2. \quad (72)$$

If someone applies the appropriate coordinate transformation of $\tilde{r} = c_{\mathcal{R}} r$ and

$$\tilde{t} = -\left(\frac{1+w_0}{w_0}\right) e^{c_\Phi} c_{\mathcal{R}}^{-w_0/(1+w_0)} e^{2w_0/(1+w_0)t}, \quad (73)$$

under this transformation rule the metric becomes

$$ds^2 = \tilde{r}^{2w_0/(1+w_0)} d\tilde{t}^2 + \frac{1}{c_{\mathcal{R}} - c_m} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2, \quad (74)$$

which is a similar static singular solution to the obtained solution for the self-similarity of the second-kind case. Analogous to the previous case, the detailed analysis in the Appendix shows that the $w(\xi)$ must remain constant, and the viscosity term should disappear.

4. Self-similarity of the infinite kind

Under the assumption of the infinite kind of self-similarity, the Einstein field equations split into terms of order $(1/t)^2$ and $(1/t)^0$. The energy density, the effective pressure, and the MSH mass are decomposed accordingly,

$$\kappa\rho = \left[\rho_0(\xi) + \frac{1}{t^2} \rho_2(\xi) \right], \quad (75)$$

$$\kappa p = \left[P_0(\xi) + \frac{1}{t^2} P_2(\xi) \right], \quad (76)$$

$$2m = \left[m_0(\xi) + \frac{1}{t^2} m_2(\xi) \right], \quad (77)$$

where the self-similarity variable $\xi = r/t$. Consequently, the definition of the MSH mass takes the form of

$$m_0 = \mathcal{R}(1 - e^{-2\Psi} \mathcal{R}^2), \quad (78)$$

$$m_2 = \mathcal{R} \mathcal{R}^2 e^{-2\Phi}, \quad (79)$$

such that spatial mass conservation equations remain similar to Eqs. (47) and (48) (with $\alpha = 1$), and the temporal part becomes

$$m'_0 = \rho_0 \mathcal{R}^2 \mathcal{R}', \quad (80)$$

$$m'_2 = \rho_2 \mathcal{R}^2 \mathcal{R}'. \quad (81)$$

The energy conservation $T_{\mu\nu;\nu} = 0$ becomes

$$(\rho_0 + P_0)\Phi' = -P'_0, \quad (82)$$

$$(\rho_2 + P_2)\Phi' = -P'_2, \quad (83)$$

$$\rho'_0 \mathcal{R} = -(\rho_0 + P_0)(\Psi' \mathcal{R} + 2\mathcal{R}'), \quad (84)$$

$$(2\rho_2 + \rho'_2) \mathcal{R} = -(\rho_2 + P_2)(\Psi' \mathcal{R} + 2\mathcal{R}'). \quad (85)$$

The (r, t) component of the Einstein field equations takes the simple form of

$$\mathcal{R}'' = \mathcal{R}'(\Phi' + \Psi'), \quad (86)$$

and the auxiliary equations are

$$\rho_2 \mathcal{R}^2 e^{2\Phi} = \mathcal{R}'(2\Psi' \mathcal{R} + \mathcal{R}'), \quad (87)$$

$$2\mathcal{R}[\mathcal{R}'' + (1 - \Phi')\mathcal{R}'] = -(\mathcal{R}'^2 + P_2 \mathcal{R}^2 e^{2\Phi}), \quad (88)$$

$$(1 - \rho_0 \mathcal{R}^2) e^{2\Psi} = 2\mathcal{R}(\mathcal{R}'' - \Psi' \mathcal{R}') + \mathcal{R}'^2, \quad (89)$$

$$(1 + P_0 \mathcal{R}^2) e^{2\Psi} = \mathcal{R}'(\mathcal{R}' + 2\Phi' \mathcal{R}). \quad (90)$$

The obtained solution for $\tilde{N}(\xi)$ in Eq. (70) still holds for this self-similarity case. We have verified in the Appendix E3 subsection that no viscous solution exists if $\rho_0 \neq 0$ since it reduces to the perfect fluid solution. Also, we demonstrated that $\rho_0 = 0$ implies that the given differential equation system does not admit a general analytic solution, but its fixed points can still be determined explicitly. The linear stability analyses resulted in the system having one singular and one stable fixed point. Around the stable fixed point, the relevant EFEs lead to the following metric form:

$$ds^2 = -\Phi_0^2 dt^2 + U_2^2 \mathcal{R}_0^2 \left(\frac{r}{t}\right)^{2U_2} dr^2 + \mathcal{R}_0^2 \left(\frac{r}{t}\right)^{2U_2} d\Omega^2, \quad (91)$$

where $U_2 = -1/8(4 + 3c_{p_2} \Phi_0^2)$ is a constant. One can make an appropriate variable transformation as $t \rightarrow t' = \Phi_0 t$ and $r \rightarrow r' = r^{U_2+1} (U_2 + 1)^{-1}$, and the metric becomes

$$ds^2 = -dt'^2 + \mathcal{R}_0^2 \left(\frac{\Phi_0}{t}\right)^{2U_2} (U_2^2 dr'^2 + r'^{2U_2} d\Omega^2), \quad (92)$$

which is equivalent with the conformally flat FRWL metric if $U_2 = 1$ and $\Sigma(\theta, k) = \theta$.

B. Parallel case

In the case when the kinematic vector field is parallel to the fluid flow, Eq. (31) becomes

$$\xi(t, r) = t \frac{\partial}{\partial t}, \quad (93)$$

and the self-similar variable ξ becomes r . Similarly, we can define the line element in the comoving frame

$$ds^2 = -t^{2(\alpha-1)} e^{2\Phi(\xi)} dt^2 + t^2 dr^2 + t^2 \mathcal{R}^2(\xi) d\Omega^2, \quad (94)$$

where α self-similarity index $\alpha \neq 0$ and $\alpha \neq 1$ for self-similarity of the second kind, and $\alpha = 0$ for self-similarity of the zeroth kind. The line element will be modified for the infinite kind, such as

$$ds^2 = -e^{2\Phi(\xi)} dt^2 + dr^2 + \mathcal{R}^2(\xi) d\Omega^2. \quad (95)$$

1. Self-similarity of the first, second, and zeroth kind

The form of Einstein's field equations, as well as in the tilted solutions, implies the form of density, effective pressure, and MSH mass, which will result in

$$\kappa\rho = t^{-2}\rho_0(\xi) + t^{-2\alpha}\rho_2(\xi), \quad (96)$$

$$\kappa P = t^{-2}P_0(\xi) + t^{-2\alpha}P_2(\xi), \quad (97)$$

$$2m = tm_0(\xi) + t^{3-2\alpha}m_2(\xi). \quad (98)$$

Since the EFEs are independently satisfied at both orders $\mathcal{O}(t^0)$ and $\mathcal{O}(t^{2-2\alpha})$, the relevant equations will be

$$(\rho_0 + P_0)\Phi' = -P'_0, \quad (\rho_2 + P_2)\Phi' = -P'_2, \quad (99)$$

$$-\rho_0 = 3P_0, \quad (2\alpha - 3)\rho_2 = 3P_2. \quad (100)$$

The Misner-Sharp-Hernandez masses take the form of

$$m_0 = \mathcal{R}(1 - \mathcal{R}'^2), \quad m_2 = \mathcal{R}^3 e^{-2\Phi}, \quad (101)$$

and the time and spatial conservation equations can be expressed as

$$m_0 = -P_0 \mathcal{R}^3, \quad (3 - 2\alpha)m_2 = -P_2 \mathcal{R}^3, \quad (102)$$

$$m'_0 = \rho_0 \mathcal{R}^2 \mathcal{R}', \quad m'_2 = \rho_2 \mathcal{R}^2 \mathcal{R}'. \quad (103)$$

Equation (23) and the auxiliary equations will take the form of

$$\mathcal{R}\Phi' = 0, \quad (104)$$

$$2\mathcal{R}''\mathcal{R} + \mathcal{R}'^2 = 1 - \rho_0 \mathcal{R}^2, \quad (105)$$

$$\mathcal{R}'^2 + 2\Phi'\mathcal{R}'\mathcal{R} = 1 + P_0 \mathcal{R}^2, \quad (106)$$

$$\rho_2 = 3e^{-2\Phi}, \quad (107)$$

$$P_2 = (2\alpha - 3)e^{-2\Phi}, \quad (108)$$

where the (\prime) denotes derivation with respect to r since $\xi = r$. It can be trivially seen from Eq. (100) that $w(\xi)$ must be constant, and thus, viscous solutions cannot exist in the case of parallel zeroth-, first-, and second-kind of self-similarity. Hence, in the asymptotic regime, such systems can be effectively described as perfect fluids.

2. Self-similarity of the infinite kind

If the $\delta = 0$, the kinematic self-similar vector field remains as it was defined in Eq. (93), as well as the invariant line element in Eq. (95). Consequently, the form of the Einstein equation will constrain the form of the matter fields, which are

$$2m = \tilde{m}(\xi), \quad \kappa\rho = \tilde{\rho}(\xi) \quad \text{and} \quad \kappa P = \tilde{P}(\xi). \quad (109)$$

It implies that the Einstein field equations [Eqs. (19)–(27)] will be

$$(\tilde{\rho} + \tilde{P})\Phi' = -P', \quad (110)$$

$$\tilde{m} = \mathcal{R}(1 - \mathcal{R}'^2), \quad (111)$$

$$\tilde{m}' = \tilde{\rho}\mathcal{R}'\mathcal{R}^2, \quad (112)$$

$$2\mathcal{R}''\mathcal{R} + \mathcal{R}'^2 = 1 - \tilde{\rho}\mathcal{R}^2, \quad (113)$$

$$\mathcal{R}'(2\Phi'\mathcal{R} + \mathcal{R}') = 1 + \tilde{P}\mathcal{R}^2. \quad (114)$$

One can see that if $w(\xi)$ does not depend on time, then the viscous term in Eq. (12) should vanish. As a result, the solution will reduce to the exact dimensionless form of the static Tolman-Oppenheimer-Volkoff equation for perfect fluid [53].

C. Orthogonal case

In the case when the kinematic vector field is orthogonal to the fluid flow and if the similarity index $\delta \neq 0$, Eq. (31) becomes

$$\xi(t, r) = \xi^\mu \frac{\partial}{\partial x^\mu} = r \frac{\partial}{\partial r}, \quad (115)$$

and the self-similar variable ξ becomes t . Similarly, we can define the line element in the comoving frame

$$ds^2 = -r^{2\alpha} dt^2 + e^{2\Psi(\xi)} dr^2 + r^2 \mathcal{R}^2(\xi) d\Omega^2. \quad (116)$$

where the δ self-similarity index is not zero. If $\delta = 0$, the KSSVF takes the form of $\xi = \partial/\partial t$ and the metric can be

expressed accordingly,

$$ds^2 = -e^{2r} dt^2 + e^{2\Psi(\xi)} dr^2 + \mathcal{R}^2(\xi) d\Omega^2. \quad (117)$$

1. Self-similarity of the first, second, and zeroth kind

The form of Einstein's field equations, as well as in the tilted solutions, implies the form of density, effective pressure, and MSH mass, which will result in

$$\kappa\rho = r^{-2}\rho_0(\xi) + r^{-2\alpha}\rho_2(\xi), \quad (118)$$

$$\kappa P = r^{-2}P_0(\xi) + r^{-2\alpha}P_2(\xi), \quad (119)$$

$$2m = rm_0(\xi) + r^{3-2\alpha}m_2(\xi). \quad (120)$$

Since the EFEs are independently satisfied at both orders $\mathcal{O}(r^0)$ and $\mathcal{O}(r^{2-2\alpha})$, the relevant energy conservation equations will be

$$-\rho'_0 = (\rho_0 + P_0)(\Psi' + 2\mathcal{R}'\mathcal{R}^{-1}), \quad (121)$$

$$-\rho'_2 = (\rho_2 + P_2)(\Psi' + 2\mathcal{R}'\mathcal{R}^{-1}), \quad (122)$$

$$\alpha\rho_0 = (2 - \alpha)P_0, \quad (123)$$

$$\alpha\rho_2 = \alpha P_2. \quad (124)$$

Similar to the parallel case, the conservation equation implies that the $w(\xi)$ effective EoS parameter should be constant; hence, the bulk viscosity should trend to zero asymptotically.

2. Self-similarity of the infinite kind

The form of the matter field is implied by the relevant dynamical equations, such as

$$\kappa\rho = e^{-2r}\rho_0(\xi) + \rho_2(\xi), \quad (125)$$

$$\kappa P = e^{-2r}P_0(\xi) + P_2(\xi), \quad (126)$$

$$m = e^{-2r}m_0(\xi) + m_2(\xi), \quad (127)$$

where $\xi = t$, similar to the previous cases, and the metric is determined in Eq. (117). In this case, there is no consistent solution for the governing equation as analyzed by Hideki Maeda *et al.* [36].

IV. DISCUSSION

We provided a detailed analysis of how the kinematic self-similarity should be applied to Landau-Lifschitz-Eckart formalism. We have shown that the kinematic conditions impose very strict constraints on the form of the bulk viscosity, limiting the dependence of the bulk

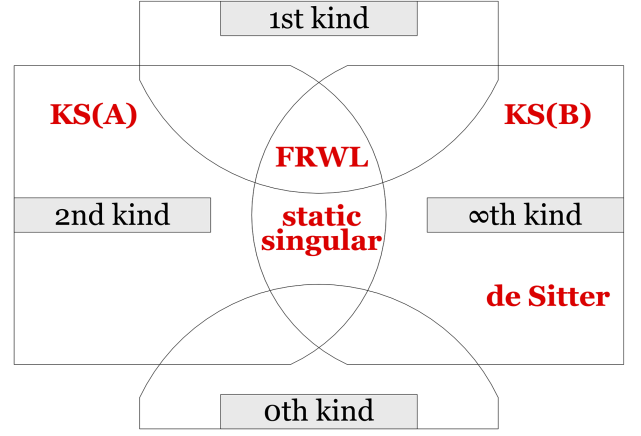


FIG. 2. Possible “tilted” kinematic self-similar solutions for spherically symmetric bulk-viscous spacetimes classified according to the known geometries. KS(A) and KS(B) indicate two types of Kantowski-Sach solutions.

viscosity to $\zeta(\rho) \propto \rho^{1/2}$ and $\zeta(\theta) \propto \theta$. Our study reveals that the mathematical conditions (provided by the KSSVF framework) are often too restrictive, inevitably leading to either vacuum solutions or perfect-fluid solutions. Thus, we proved that if the KSSVF is either orthogonal or parallel to the fluid flow, it does not lead to solutions that involve viscous flows. In contrast to the classes discussed above, the extension to bulk viscous flows can be consistently carried out in the case of “tilted” solutions. Their classification, along with the various possible solutions, is shown in Fig. 2. These results are in complete agreement with previous studies conducted on perfect fluid cases [34,36,37]. The obtained solutions are identified with well-known cosmological spacetimes, including the Friedmann-Robertson-Walker-Lemaître models [39], the Einstein-de Sitter universe [54], and the Kantowski-Sachs solutions [50]. Building on the well-established similarity hypothesis introduced by Carr, which posits that spherically symmetric flows tend toward self-similar configurations at late times, our analysis of viscous models underscores the utility of self-similar solutions in clarifying their asymptotic properties. However, when the ultimate (large-scale) boundary conditions influence the evolution, the notion of self-similarity as a true asymptotic state must be modified, and the resulting solutions should rather be regarded as intermediate asymptotes in the sense of Barenblatt [11,55].

A comparison of the asymptotic dynamics of viscous systems with those of perfect fluids indicates that the two systems display analogous behavior. In astrophysics, this implies that viscous effects may be challenging to constrain theoretically since, in the context of kinematic self-similarity, both types of flows tend to similar spacetime configurations at late times, making it difficult to distinguish the influence of viscosity solely on the asymptotic evolution of the system. This phenomenon is particularly

relevant in cosmological contexts, during gravitational collapse [48], and in the formation of compact objects [56], where the long-term behavior of the fluid plays a crucial role. This trend is further supported by studies that find asymptotically perfect, spherically symmetric solutions of relativistic dissipative hydrodynamics in flat space using the Israel-Stewart theory, demonstrating that dissipation may not alter the late-time evolution qualitatively [57].

V. OUTLOOK

A natural extension of this analysis is to move beyond the first-order Eckart-Landau-Lifshitz flow by adopting a second-order theory of relativistic dissipation, such as Müller-Israel-Stewart theory, which overcomes the well-known causality and instability issues of first-order theories. Extending the current classification scheme into this causal regime could uncover new classes of physically admissible solutions relevant to cosmology [58], gravitational collapse, and the formation of compact objects. Similarly, analyzing shear-driven anisotropic stresses within the same frame would complete the description of viscous effects, offering a unified view of relativistic dissipation in strongly gravitating systems.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: NOTATION AND CONVENTIONS

Our paper used the $c = 1$, $h = 1$, $G = 1$ unit system and $\kappa = 8\pi$. For arbitrary contravariant coordinates x^μ , where Greek letters run from 0 to 3, we have used the $g_{\mu\nu}$ covariant metric tensor with the $(-, +, +, +)$ signature to describe the four-dimensional spacetime. The semicolon $X_{\mu;\nu}$ denotes the covariant derivative. Also, the prime ($'$) and (\cdot) represent the partial derivative by spatial and time coordinates, respectively. The $d\Omega^2 = d\theta^2 + \Sigma^2(k, \theta)d\phi^2$.

APPENDIX B: FOUR-VELOCITY DECOMPOSITION

The decomposition of the covariant derivative of the four-velocity is

$$u_{\mu;\nu} = h_{\mu}^{\rho} h_{\nu}^{\kappa} u_{\rho;\kappa} + \dot{u}_{\mu} u_{\nu} = b_{\mu\nu} + \dot{u}_{\mu} u_{\nu}, \quad (\text{B1})$$

using the definition of the relevant covariant derivative [59]. The $b_{\mu\nu}$ can be decomposed into a symmetric and skew-symmetric part

$$b_{\mu\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu}, \quad (\text{B2})$$

where

$$\sigma_{\mu\nu} = \theta_{(\mu\nu)} = h_{(\mu}^{\rho} h_{\nu)}^{\kappa} u_{\rho;\kappa}, \quad (\text{B3})$$

and

$$\omega_{\mu\nu} = \omega_{[\mu\nu]} = h_{[\mu}^{\rho} h_{\nu]}^{\kappa} u_{\rho;\kappa}. \quad (\text{B4})$$

The vorticity tensor, $\omega_{\mu\nu}$, is the skew-symmetric part. It determines the rigid rotation of the flow with respect to a local inertial rest frame. The symmetric part $\theta_{\mu\nu}$ is traditionally decomposed further into traceless and trace parts:

$$\theta_{\mu\nu} = \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu}, \quad (\text{B5})$$

so that

$$\theta_{\mu}^{\mu} = 0 \quad \text{and} \quad \theta = u^{\mu}_{;\mu}. \quad (\text{B6})$$

We are also going to introduce the magnitude of the vorticity and the symmetric part,

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} \quad \text{and} \quad \omega^2 = \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}. \quad (\text{B7})$$

Finally, the acceleration vector is defined in Eq. (B1) in the following way:

$$\dot{u}^{\mu} = u^{\mu}_{;\nu} u^{\nu}. \quad (\text{B8})$$

APPENDIX C: LIE DERIVATIVE OF THE PRESSURE, FOUR-VELOCITY, AND ENERGY DENSITY IN THE CASE OF THE VISCOUS ENERGY TENSOR

Let u^{μ} be a timeline vector field normalized in such way $u_{\mu}u^{\mu} = -1$. The Lie derivative \mathcal{L}_{ξ} along a homothetic vector field ξ can be expressed in the following way:

$$\begin{aligned}
0 &= \mathcal{L}_\xi(u_\mu u^\mu) = \mathcal{L}_\xi(g_{\mu\nu} u^\mu u^\nu) \\
&= 2u_\mu \mathcal{L}_\xi u^\mu + u^\mu u^\nu \mathcal{L}_\xi g_{\mu\nu} = 2u_\mu \mathcal{L}_\xi u^\mu - 2\alpha. \quad (\text{C1})
\end{aligned}$$

Consequently, the Lie derivative of the covariant and contravariant timeline vector field u_μ should be

$$\mathcal{L}_\xi u^\mu = -u^\mu - v^\mu \quad \text{and} \quad \mathcal{L}_\xi u_\mu = u_\mu - v_\mu, \quad (\text{C2})$$

where u_μ and v_μ are orthogonal to each other. According to the Eq. (4) and the form of the viscous energy-momentum tensor Eq. (10), we can derive the following relation:

$$\begin{aligned}
0 &= \mathcal{L}_\xi T_{\mu\nu} \\
&= \mathcal{L}_\xi[(\rho + P)u_\mu u_\nu + P g_{\mu\nu} + 2\eta\sigma_{\mu\nu}] \\
&= (\mathcal{L}_\xi \rho + \mathcal{L}_\xi P)u_\mu u_\nu + (\mathcal{L}_\xi P)g_{\mu\nu} \\
&\quad + (\rho + P)\mathcal{L}_\xi(u_\mu u_\nu) + P\mathcal{L}_\xi g_{\mu\nu} + 2\eta\mathcal{L}_\xi \sigma_{\mu\nu} \\
&= (\mathcal{L}_\xi \rho + \mathcal{L}_\xi P)u_\mu u_\nu + (\mathcal{L}_\xi P)g_{\mu\nu} + (\rho + P)\mathcal{L}_\xi(u_\mu u_\nu) \\
&\quad + P\mathcal{L}_\xi g_{\mu\nu} + 2\eta\mathcal{L}_\xi \sigma_{\mu\nu} + \mathcal{L}_\xi \Pi(g_{\mu\nu} + u_\mu u_\nu) + \Pi\mathcal{L}_\xi g_{\mu\nu}. \quad (\text{C3})
\end{aligned}$$

If we use the formula for the projection tensor $h_{\mu\nu}$, then

$$\begin{aligned}
0 &= (\mathcal{L}_\xi \rho + 2\alpha\rho)u_\mu u_\nu + (\mathcal{L}_\xi + 2\alpha)[p + \Pi]h_{\mu\nu} \\
&\quad - (\rho + p + \Pi)(u_\mu v_\nu - u_\nu v_\mu) + 2\eta\mathcal{L}_\xi \sigma_{\mu\nu}. \quad (\text{C4})
\end{aligned}$$

For the arbitrary vector field V^μ and the ξ homothetic vector field the following identity holds

$$\mathcal{L}_\xi(\nabla_\mu V^\mu) \stackrel{!}{=} \nabla_\mu(\mathcal{L}_\xi V^\mu). \quad (\text{C5})$$

The general commutation relation is

$$\begin{aligned}
[\nabla_\mu, \mathcal{L}_\xi]V^\mu &= \nabla_\mu(\mathcal{L}_\xi V^\mu) - \mathcal{L}_\xi(\nabla_\mu V^\mu) \\
&= (\nabla_\mu \xi^\nu)(\nabla_\nu V^\mu) - (\nabla_\nu V^\mu)(\nabla_\mu \xi^\nu) - V^\nu \nabla_\mu \nabla_\nu \xi^\mu. \quad (\text{C6})
\end{aligned}$$

Applying the decomposition of $\nabla_\mu \xi^\nu = 2\alpha\delta_\mu^\nu + \omega_{\mu\nu}$, where $\omega_{\mu\nu}$ is antisymmetric, it becomes trivial to show that the first and second terms on the r.h.s will vanish. Using the Ricci identity, the third term transforms as

$$\nabla_\mu \nabla_\nu \xi^\mu = -R_{\mu\nu} \xi^\mu \Rightarrow -V^\nu \nabla_\mu \nabla_\nu \xi^\mu = V^\nu R_{\mu\nu} \xi^\mu, \quad (\text{C7})$$

and it will vanish if ξ is a proper HVF [21,60]. By contracting Eq. (C4) in different ways and setting $\sigma_{\mu\nu}$ to zero, it follows that

$$u^\mu u^\nu \mathcal{L}_\xi T_{\mu\nu} = 0 \rightarrow \mathcal{L}_\xi \rho = -2\alpha\rho, \quad (\text{C8})$$

$$h_\sigma^\mu h_\rho^\nu \mathcal{L}_\xi T_{\mu\nu} = 0 \rightarrow \mathcal{L}_\xi P = -2\alpha P, \quad (\text{C9})$$

$$u^\mu h_\rho^\nu \mathcal{L}_\xi T_{\mu\nu} = 0 \rightarrow (\rho + P)v_\sigma = 0. \quad (\text{C10})$$

The last relation of Eq. (C10) implies that if $w(\xi) \neq -1$ and a nonvacuum solution is considered, the v_σ must vanish, and then

$$\mathcal{L}_\xi u^\mu = -u^\mu \quad \text{and} \quad \mathcal{L}_\xi u_\mu = u_\mu. \quad (\text{C11})$$

APPENDIX D: GENERAL SPHERICALLY SYMMETRIC SPACETIME

From the line element of Eq. (8) of the most general spherically symmetric spacetime, the nonzero elements of the Christoffel symbol are

$$\begin{aligned}
\Gamma_{tt}^t &= \dot{\phi}, \quad \Gamma_{tr}^t = \Gamma_{rt}^t = \phi' \\
\Gamma_{rr}^r &= \dot{\psi}e^{2\psi-2\phi}, \quad \Gamma_{\varphi\varphi}^r = \Sigma(k, \theta)^2 \Gamma_{\theta\theta}^r, \quad \Gamma_{\theta\theta}^r = e^{-2\phi} R \dot{R}, \\
\Gamma_{tt}^r &= e^{2\phi-2\psi} \phi', \quad \Gamma_{tr}^r = \Gamma_{rt}^r = \dot{\psi}, \\
\Gamma_{rr}^r &= \psi', \quad \Gamma_{\theta\theta}^r = -e^{-2\psi} R R', \quad \Gamma_{\varphi\varphi}^r = \Sigma^2(k, \theta) \Gamma_{\theta\theta}^r, \\
\Gamma_{t\theta}^\theta &= \Gamma_{\theta t}^\theta = \Gamma_{r\varphi}^\theta = \Gamma_{\varphi r}^\theta = \dot{R} R^{-1}, \\
\Gamma_{r\theta}^\theta &= \Gamma_{\theta r}^\theta = \Gamma_{r\varphi}^\theta = \Gamma_{\varphi r}^\theta = R' R^{-1}, \\
\Gamma_{\varphi\varphi}^\theta &= \Sigma(k, \theta) \Sigma'(k, \theta), \quad \Gamma_{\theta\varphi}^\theta = \Gamma_{\varphi\theta}^\theta = \Sigma^{-1}(k, \theta). \quad (\text{D1})
\end{aligned}$$

The nonzero components of the Ricci-tensor are

$$\begin{aligned}
R_t^t &= R^{-1} \left\{ e^{-2\Phi} (R(\ddot{\Psi} + \ddot{\Psi}^2 - \dot{\Phi}\dot{\Psi}) + 2\ddot{R} - 2\dot{R}\dot{\Phi}) \right. \\
&\quad \left. - e^{-2\Psi} (R\Phi'' + \Phi'(R(\Phi' - \Psi') + 2R')) \right\}, \quad (\text{D2})
\end{aligned}$$

$$R_r^r = 2e^{-2\Psi} R^{-1} (-\ddot{R}' + R'\dot{\Psi} + \dot{R}\Phi'), \quad (\text{D3})$$

$$R_t^r = 2e^{-2\Phi} R^{-1} (-\dot{R}' + R'\dot{\Psi} + \dot{R}\Phi'), \quad (\text{D4})$$

$$\begin{aligned}
R_r^r &= e^{-2\Psi} [R^{-1} (e^{2(\Psi-\Phi)} (R\ddot{\Psi} + \dot{\Psi}(R(\dot{\Psi} - \dot{\Phi}) + 2\dot{R})) \\
&\quad - R\Phi'' - 2R'' + 2R'\Psi') + \Phi'(\Psi' - \Phi')], \quad (\text{D5})
\end{aligned}$$

$$\begin{aligned}
R_\theta^\theta &= \frac{e^{-2(\Psi+\Phi)}}{R^2} \left[e^{2\Psi} (e^{2\Phi} + R\ddot{R} + R\dot{R}(\dot{\Psi} - \dot{\Phi}) + \dot{R}^2) \right. \\
&\quad \left. - e^{2\Phi} (RR'' + RR'(\dot{\Phi} - \dot{\Psi}) + (R')^2) \right], \quad (\text{D6})
\end{aligned}$$

$$R_\phi^\phi = \Sigma^2(k, \theta) R_\theta^\theta, \quad (\text{D7})$$

By taking the trace of the Ricci tensor with respect to the metric tensor, the Ricci scalar is obtained, such as

$$\begin{aligned}
R &= \frac{e^{-2(\Psi+\Phi)}}{R^2} \left\{ 2e^{2\Psi} [e^{2\Phi} + 2R\dot{R}(\dot{\Psi} - \dot{\Phi}) + \dot{R}^2 + 2R\ddot{R} \right. \\
&\quad \left. + R^2(\ddot{\Psi}^2 + \ddot{\Psi} - \dot{\Psi}\dot{\Phi})] - 2e^{2\Phi} (2RR'(\Phi' - \Psi') \right. \\
&\quad \left. + R(2R'' + R(-\Psi'\Phi' + \Phi'^2 + \Phi'')) + R^2] \right\}. \quad (\text{D8})
\end{aligned}$$

APPENDIX E: IN-DEPTH ANALYSIS AND DERIVATION OF THE TILTED SELF-SIMILAR SOLUTIONS

1. Self-similarity of second-kind

First, we eliminated the second derivative \mathcal{R}'' using Eq. (49), and we find that

$$\mathcal{R}'' = \mathcal{R}'\Phi' + (\mathcal{R} + \mathcal{R}')\Psi' - \mathcal{R}'. \quad (\text{E1})$$

Substituting this formula into Eq. (53), followed by simplification, the auxiliary equation yields

$$2\mathcal{R}\mathcal{R}'\Phi' + 2\mathcal{R}\mathcal{R}' = -\mathcal{R}'^2 - \mathcal{R}^2 + e^{2\Psi}(1 - \rho_0\mathcal{R}^2). \quad (\text{E2})$$

Subtracting Eq. (E2) from Eq. (55) we arrive at the following equation:

$$[\mathcal{R}^2 + \mathcal{R}'^2 + 2\Phi'\mathcal{R}(\mathcal{R} + \mathcal{R}')] - [2\mathcal{R}\mathcal{R}'\Phi'] \\ = (1 + P_0\mathcal{R}^2)e^{2\Psi} + [\mathcal{R}'^2\mathcal{R}^2 - e^{2\Psi}(1 - \rho_0\mathcal{R}^2)]. \quad (\text{E3})$$

Canceling the common terms on both sides, we finally obtain a constraint equation between the pressure, the density, and metric functions,

$$2\Phi' = (\rho_0 + P_0)e^{2\Psi}. \quad (\text{E4})$$

From the relation between Eqs. (38) and (39), the effective pressure is assumed to be $P_0 = w(\xi)\rho_0$ and $P_2 = w(\xi)\rho_2$. Substituting these into Eqs. (41) and (42) will result in

$$(1 + w)\Phi' = \left(2 - \frac{\rho'_0}{\rho_0}\right)w - w', \quad (\text{E5})$$

$$(1 + w)\Phi' = -\left(w\frac{\rho'_2}{\rho_2} + w'\right). \quad (\text{E6})$$

By forming the ratio of these equations, one obtains

$$\frac{\rho'_0}{\rho_0} = 2 + \frac{\rho'_2}{\rho_2}. \quad (\text{E7})$$

Similarly, Eqs. (43) and (44) will lead to the relation of

$$\frac{\rho'_0}{\rho_0} = 2\alpha + \frac{\rho'_2}{\rho_2}. \quad (\text{E8})$$

This leads us to $\alpha = 1$, if $\rho_0\rho_2 \neq 0$ and concludes that the self-similar solution of the second kind can exist only if $\rho_0\rho_2 = 0$, similar to the perfect fluid case [36].

The $\rho_0 = 0$ case: If we assume that ρ_0 and, consequently, $P_0 = 0$, from Eq. (47) and from Eq. (45) we obtained that $m_0 = 0$. Neglecting the trivial case, Eq. (50) leads to the constraint of

$$(\mathcal{R} + \mathcal{R}') = \pm e^\Psi. \quad (\text{E9})$$

Likewise, Eq. (E4) implies that the metric function of $\Phi = \ln \phi_0$ is constant. If we substitute this into Eq. (42), it will result that P_2 is constant and

$$w'\rho_2 + \rho'_2w = 0 \quad \Rightarrow \quad w(\xi)\rho_2(\xi) = \xi_0. \quad (\text{E10})$$

Combining the remaining equations of Eqs. (46), (48), and (51), one can eliminate the m_2 and m'_2 terms and obtain the nonlinear evolution equation of

$$\rho_2 = U_0 \frac{U(\xi)}{1 + U(\xi)}, \quad \Rightarrow \quad U_0 := \left(\xi_0 + \frac{3 - 2\alpha}{\alpha^2 \Phi_0^2}\right), \quad (\text{E11})$$

where $U(\xi) = \mathcal{R}'/\mathcal{R}$. Substituting the obtained expression for ρ_2 into Eq. (44) will result in

$$\frac{2\alpha U}{1 + U} + \frac{U'}{(1 + U)^2} = \left[\frac{\xi_0}{U_0} - \frac{U}{1 + U}\right](\Psi' + 2U). \quad (\text{E12})$$

This can be transformed into a Riccati-type equation by substituting the following relation obtained from Eq. (E9):

$$\Psi' = U + \frac{U'}{1 + U}. \quad (\text{E13})$$

The closed-form solution for $U(\xi)$ is

$$U(\xi) = \frac{C_U}{3 + (1 - \tilde{U}_0)^{C_U - 1} \xi^{-C_U}}, \quad (\text{E14})$$

where $\tilde{U}_0 = \xi_0/U_0$, the $C_U = (3\tilde{U}_0 - 2\alpha)/(1 - \tilde{U}_0)$, and the integration constant is neglected. Applying the definition of $U(\xi)$ and integrating over the self-similarity variable, one can express the $\mathcal{R}(\xi)$ as a function of ξ ,

$$\mathcal{R} = \mathcal{R}_0(3(1 - \tilde{U}_0)\xi^{C_U} + (1 - \tilde{U}_0)^{C_U})^{1/3}. \quad (\text{E15})$$

Applying this formula in Eq. (E9) one can solve the ordinary differential equation for the $\Psi(\xi)$ function

$$\Psi(\xi) = \ln(\mathcal{R}_0 \xi^{C_U/3} (3 - 2\alpha + (1 - \tilde{U}_0)^{C_U} \xi^{-C_U}) \\ \times (3(1 - \tilde{U}_0) + (1 - \tilde{U}_0)^{C_U} \xi^{-C_U})^{-2/3}), \quad (\text{E16})$$

where \mathcal{R}_0 is some integration constant. Equation (E9) yields a complex solution in addition to Eq. (E16). However, this complex solution is typically discarded in physical applications. The expression of Eq. (E10) and the definition of the effective $w(\xi)$ reveal that there are only two possibilities: either $\zeta \rightarrow 0$, which reduces to the perfect fluid case, or $w_0, U_0 \rightarrow 0$, corresponding to the vacuum case. However, the $\zeta \propto \theta$ bulk viscosity form is consistent

with the obtained solution if w_0 vanishes, the $\xi_0/\zeta U_0$ is set to unity, and the $(1 - U_0)^3 \stackrel{!}{=} 9$.

The $\rho_2 = 0$ case: In the second case, where $\rho_0 \neq 0$, the ρ_2 and P_2 are zero. From Eqs. (43) and (52), one can obtain the equation of

$$\frac{1}{1 + w(\xi)} \frac{\rho'_0}{\rho_0} = -\frac{3}{2} \frac{\mathcal{R}'}{\mathcal{R}} \quad (\text{E17})$$

Since the $P_2 = 0$, the Eq. (54) becomes more trivial,

$$\mathcal{R}'' - \Phi' \mathcal{R}' = -\frac{1}{2} \left(\frac{\mathcal{R}'}{\mathcal{R}} + 2\alpha \right) \mathcal{R}'. \quad (\text{E18})$$

Using Eq. (49) we can eliminate the $\mathcal{R}'' - \Phi' \mathcal{R}'$ from Eq. (E18). We can obtain an expression for Ψ' , and substituting this formula to Eq. (43), we have

$$(\alpha - 1)U(\xi) + \frac{1}{2}U^2(\xi) - [1 + U(\xi)] \left(2U(\xi) + \frac{1}{1 + w(\xi)} V(\xi) \right) = 0, \quad (\text{E19})$$

where $V(\xi) = \rho'_0/\rho_0$. Applying the constraint from Eq. (E17), we have

$$(2\alpha - 9)U(\xi) - 6U^2(\xi) = 0. \quad (\text{E20})$$

Solving this nonlinear equation provides two distinct solutions. The trivial solution corresponds to the case where $U(\xi) = 0$, and \mathcal{R} remains fixed at the constant value \mathcal{R}_0 . The nontrivial solution is

$$U(\xi) = \frac{1}{3} \left(\alpha - \frac{9}{2} \right) \Rightarrow \mathcal{R}(\xi) = \mathcal{R}_0 \xi^{\lambda_{0,r}}, \quad (\text{E21})$$

where $\lambda_{0,r} = (2\alpha - 9)/6$ is some constant.

(A) Applying the nontrivial solution the Eqs. (E18) and (52) becomes

$$\Phi(\xi) = c_\Phi \ln \xi \quad \text{and} \quad \Psi(\xi) = -\frac{1}{2} \lambda_{0,r} \ln \xi, \quad (\text{E22})$$

where $c_\Phi = 3(2\alpha - 3)/4$. Hence, we can use Eq. (E17) to find ρ_0 if the bulk viscosity parameter depends on the density $\sim \rho^{1/2}$,

$$\rho_0(\xi) = \left(\frac{\frac{3}{4} \zeta \lambda_{0,r}^2}{2(-c_\Phi + \frac{3}{4} \lambda_{0,r}(1 + w_0))} \xi^{-c_0} + c_{\rho,0} \xi^{-\frac{3}{4} \lambda_{0,r}(1 + w_0)} \right)^2, \quad (\text{E23})$$

where $c_{\rho,0}$ is some integration constant. Finally, one can obtain the following formula for the P_0 :

$$P_0(\xi) = w_0 \left(\frac{\frac{3}{4} \zeta \lambda_{0,r}^2}{2(-c_\Phi + \frac{3}{4} \lambda_{0,r}(1 + w_0))} \xi^{-c_0} + c_{\rho,0} \xi^{-\frac{3}{4} \lambda_{0,r}(1 + w_0)} \right)^2 - \frac{1}{2} \zeta \lambda_{0,r} \xi^{-c_0} \left(\frac{\frac{3}{4} \zeta \lambda_{0,r}^2}{2(-c_\Phi + \frac{3}{4} \lambda_{0,r}(1 + w_0))} \xi^{-c_0} + c_{\rho,0} \xi^{-\frac{3}{4} \lambda_{0,r}(1 + w_0)} \right). \quad (\text{E24})$$

Consequently, from the definition of the MSH mass Eqs. (50) and (51), we obtain the formula for m_0 and m_2

$$m_0 = \mathcal{R}_0 \xi^{\lambda_{\rho,0}} [1 - (1 + \lambda_{\rho,0})^2 \mathcal{R}_0^2 \xi^{4\lambda_{\rho,0}}], \quad (\text{E25})$$

and the m_2 will vanish according to the Eqs. (46) and (48).

(B) Applying the trivial solution of $U(\xi)$, the radial term is $\mathcal{R}(\xi) = \mathcal{R}_0$. Hence, $\Psi = \ln \psi_0$ and $m_0 = \mathcal{R}_0(1 - \mathcal{R}_0^2/\psi_0^2)$ also became constant based on the obtained solutions from Eqs. (49) and (50). Equation (45) implies that ρ_0 must be constant

$$\rho_0 = \frac{\mathcal{R}_0^2 - \psi_0^2}{\mathcal{R}_0^2 \psi_0^2}. \quad (\text{E26})$$

It is apparent from the definition of the $w(\xi)$ that the bulk viscosity part will vanish and become a constant of w_0 , and the P_0 will be a constant. It is evident from Eq. (41) that $\Phi(\xi)$ can be expressed in terms of ξ and w_0 in the following way:

$$\Phi(\xi) = \frac{2w_0}{(1 + w_0)} \ln \xi. \quad (\text{E27})$$

2. Self-similar of the zeroth kind

Since the governing equations do not change drastically if we impose the 0th kind of self-similarity, the derived constraint for Φ described in Eq. (E4) still holds. Also, the obtained condition of $\rho_0(\xi)\rho_2(\xi) = 0$, required for the existence of a solution, remains valid.

First, we require the ρ_0 to be zero, and since P_0 and also m_0 vanish trivially, we apply the EoS and Eq. (68). Hence, Eq. (E4) suggest that $\Phi = \ln \phi_0$ to be constant. Since $m_0 = 0$, the definition of MSH mass indicates that

$$(\mathcal{R} + \mathcal{R}') = \pm e^\Psi, \quad (\text{E28})$$

if we neglect the solution, where $\mathcal{R}(\xi) = 0$. One can express the $\Psi(\xi)$ variable and substituting it into Eq. (49) will result in

$$\mathcal{R}(\xi) = \mathcal{R}_0 \xi \quad \text{and} \quad \Psi(\xi) = \ln(2\mathcal{R}_0 \xi). \quad (\text{E29})$$

The minus sign in Eq. (E28) would correspond to $\mathcal{R}_0 < 0$, which leads to a complex solution, and therefore, it is nonphysical, thus disregarded. Consequently, the definition of the second-order MSH mass of Eq. (51) (with $\alpha = 1$)

implies that $m_2(\xi) = \mathcal{R}_0^3 \Phi_0^2 \xi^3$. Also, Eq. (69) suggests that the pressure equals $P_2 = -3\Phi_0^2$. That indicates that ρ_2 is also constant, and the solution is equivalent to the perfect fluid case. Also, Eq. (69) suggests that the pressure equals $P_2 = -3\Phi_0^2$. That indicates that ρ_2 is also constant, and the solution is equivalent to the perfect fluid case.

Secondly, we have the case where $\rho_2 = 0$ and consequently P_2 vanishes. Equations (51) and (46) suggest that m_2 should also vanish. If we take the 0th self-similar form of Eq. (E18), it becomes

$$\mathcal{R}'' - \Phi' \mathcal{R}' = -\frac{1}{2} \frac{\mathcal{R}'^2}{\mathcal{R}}, \quad (\text{E30})$$

and similarly Eq. (E19) transforms as

$$3U(\xi) + \frac{3}{2}U^2(\xi) + (1 + U(\xi)) \left(\frac{1}{1 + w(\xi)} V(\xi) \right) = 0, \quad (\text{E31})$$

where $U(\xi) = \mathcal{R}'/\mathcal{R}$ and $V(\xi) = \rho_0'/\rho_0$, respectively. Also, similar to the 2nd kind of self-similarity case we have the

$$\frac{1}{1 + w(\xi)} V(\xi) = -\frac{3}{2} U(\xi). \quad (\text{E32})$$

It is trivial to see from Eqs. (E31) and (E32), which lead to $U(\xi)$ being zero and $\mathcal{R}(\xi) = c_{\mathcal{R}}$. Consequently, $\rho_0 = c_{\rho,0}$ will be constant. Substituting the obtained solutions into Eq. (68) will result in $m_0(\xi)$ being constant. Also, the Ψ can be determined from the MSH mass definition of Eq. (50),

$$\Psi(\xi) = -\frac{1}{2} \ln \left(\frac{c_{\mathcal{R}} - c_m}{c_{\mathcal{R}}^2} \right). \quad (\text{E33})$$

Note that these constants are not fully independent, such as Eq. (50) implying that $c_m \stackrel{\dagger}{=} -c_{\rho,0} c_{\mathcal{R}}^3$. The last equation of Eq. (41) reveals that $w(\xi)$ must be constant, and the viscosity term will vanish. The obtained $\Phi(\xi)$ metric function is

$$\Phi(\xi) = \frac{2w_0}{w_0 + 1} \ln \xi + c_{\Phi}. \quad (\text{E34})$$

3. Self-similarity of the infinite kind

The form of the density and effective pressure in Eqs. (75) and (76) imply that the $P_0 = w(\xi)\rho_0$, and $P_2 = w(\xi)\rho_2$ is similarly to previous cases. First, we can express the \mathcal{R}'' and Ψ' from Eqs. (86) and (89) and substitute them into Eq. (90), and we arrive at the result of

$$(1 + w(\xi))\rho_0 \mathcal{R}^2 e^{2\Psi} = 0. \quad (\text{E35})$$

The energy conservation of Eqs. (82) and (83) imply that $\rho_0(\xi) = C\rho_2(\xi)$, and consequently, from Eq. (E35), we have

$$w(\xi) = -1 \quad \text{or} \quad \rho_0 = 0 \Rightarrow C \stackrel{\dagger}{=} 0, \quad (\text{E36})$$

since we are not interested in vacuum solutions. One can notice that the condition $w(\xi) = -1$ reduces the problem to the perfect fluid solution, and hence, no bulk viscous solution exists. If $\rho_0 = 0$, then trivially from Eq. (80) the mass m_0 is constant and chosen to be zero. Let us assume that $e^{\Phi} = \Phi_0$, then from Eq. (83) we obtain that

$$P_2 = c_{P_2} \quad \text{and} \quad w(\xi)\rho_2(\xi) = c_{P_2}. \quad (\text{E37})$$

We repeat the process from the first step and express the \mathcal{R}'' and Ψ' from Eqs. (86) and (87) and substitute them into Eq. (88), and

$$\rho_2 = -2\Phi_0^{-2} \frac{\mathcal{R}'}{\mathcal{R}}. \quad (\text{E38})$$

From the definition of the MSH mass, we have $\mathcal{R}' = \pm e^{\Psi}$, and substituting this and ρ_2 from Eq. (E38) into Eq. (85), we have obtained

$$-4\Phi_0^{-2}U - 2\Phi_0^{-2}(U' + U^2) = (2\Phi_0^{-2}U + c_{P_2}) \left(3U + \frac{U'}{U} \right), \quad (\text{E39})$$

where $U = \mathcal{R}'/\mathcal{R}$. This differential equation cannot be solved by analytical terms. However, we could assume that U' vanishes, and the ordinary differential equation reduces to an algebraic equation of

$$U(4 + 3c_{P_2}\Phi_0^2 + 8U) = 0, \quad (\text{E40})$$

which yields two constant solutions,

$$U_1 = 0 \quad \text{and} \quad U_2 = -\frac{1}{8}(4 + 3c_{P_2}\Phi_0^2). \quad (\text{E41})$$

To assess the stability of the constant solutions to the differential equation when U' is infinitesimally small, we perform a linear stability analysis $U(\ln \xi) = U_0 + \delta(\ln \xi)$ around each U_0 fixed point. One can clearly see that the term of U'/U from Eq. (E39) diverges at $U_1 = 0$, which makes it a singular point and consequently unstable. However, the linear stability analysis indicates that the U_2 fixed point is regular and generally stable if $c_{P_2} > -4/3\Phi_0^{-2}$. Applying the definition of U , we obtain the following expressions for \mathcal{R} and Ψ :

$$\mathcal{R}(\xi) = \mathcal{R}_0 \xi^{U_2} \quad \text{and} \quad e^{\Psi} = \pm U_2 \mathcal{R}_0 \xi^{U_2}. \quad (\text{E42})$$

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