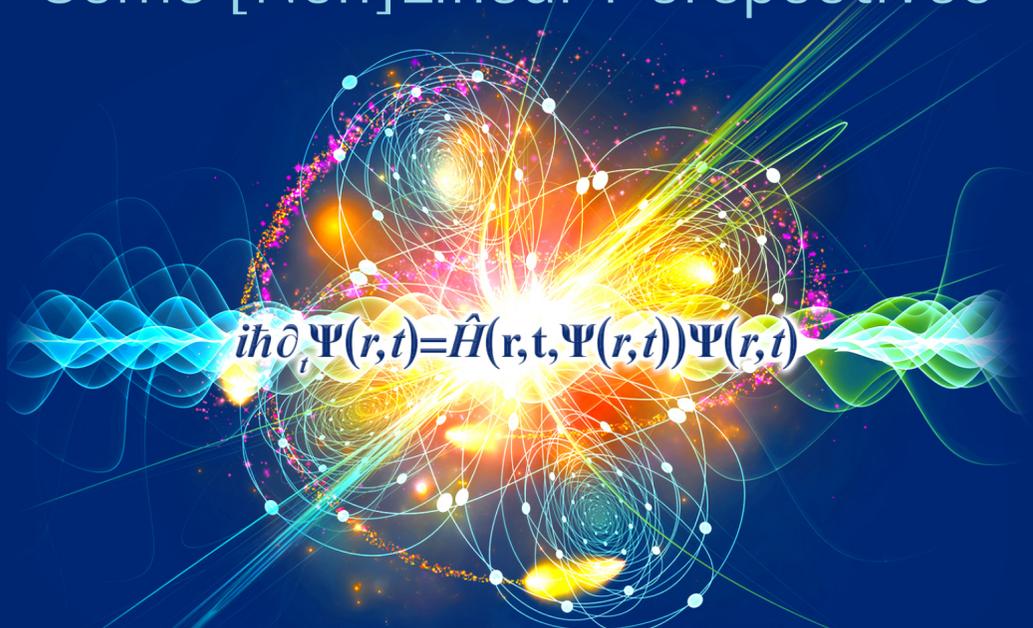


MATHEMATICS RESEARCH DEVELOPMENTS

Understanding the Schrödinger Equation

Some [Non]Linear Perspectives


$$i\hbar\partial_t\Psi(r,t)=\hat{H}(r,t,\Psi(r,t))\Psi(r,t)$$

Valentino A. Simpao • Hunter C. Little
Editors

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**UNDERSTANDING THE
SCHRÖDINGER EQUATION
SOME [NON]LINEAR PERSPECTIVES**

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**VALENTINO A. SIMPAO
AND
HUNTER C. LITTLE
EDITORS**



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FOREWORD

The book contains various approaches to the Schrödinger equation (SE) as a fundamental equation of quantum mechanics.

In Chapter 1, a new pedagogical paradigm is proposed which allows one to understand quantum mechanics as an extension of probability theory; its purpose is providing alternative methods to understand the Schrödinger equation.

Chapter 2 is devoted to the derivation of SE from the classical Hamiltonian by some procedure of second quantization.

In Chapters 3–5, the authors consider the nonlinear SE with many applications: from nonlinear waves in deep water to formation of a cosmogonical body, surface gravity waves, superconductivity and nonlinear optics.

The goal of Chapter 6 is to establish the connection of Schrödinger, Madelung and Gross-Pitaevskii equations.

Chapter 7, “Paradigm of infinite dimensional phase space,” describes the deep connection between SE and the infinite chain of equations for distribution functions of high-order kinematical values (Vlasov chain). The authors formulate the principles which allow one to combine and treat in unified form the physics of classical, statistical and quantum mechanical phenomena. And, finally, in Chapter 8 it is shown that SE can be mathematically derived from Hamilton’s equation if one uses the

metaplectic representation of canonical transformations. All that makes the book interesting for a wide community of physicists.

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PREFACE

The current offering from Nova Science Publishers titled *Understanding the Schrödinger Equation: Some [Non]Linear Perspectives* is a collection of selectively invited manuscripts from some of the world's leading workers in quantum dynamics; particularly as concerning Schrödinger's wavefunction formalism. The work is dedicated to providing an "illustrative sketch" of a few of the numerous and sundry aspects of the Schrödinger equation; ranging from a new pedagogical teaching approach, to technical applications and foundational considerations. Towards this end, the work is generally of a theoretical nature; expounding various physical aspects of both linear and nonlinear Schrödinger systems and their attendant mathematical developments.

Specifically, the book contains:

A chapter meant to give a new pedagogical paradigm for teaching an understanding of quantum mechanics, via the Schrödinger equation as an extension of probability theory;

A chapter addressing the Schrödinger equation written in the second quantization formalism, derived from first principles; towards a deeper understanding of classical-quantum correspondence;

A chapter discussing the connection between the Schrödinger equation and one of the most intuitive research fields in classical mechanics: the theory of nonlinear water waves;

A chapter which investigates wave solutions of the generalized nonlinear time-dependent Schrödinger-like equation describing a cosmogonical body formation;

A chapter addressing the nonlinear Schrödinger equation: a mathematical model with its wide-ranging applications and analytical results;

A chapter investigating analytical self-similar and traveling-wave solutions of the Madelung equations obtained from the Schrödinger equation;

A chapter which puts forth a novel paradigm of infinite dimensional quantum phase space extension of the Schrödinger equation;

A chapter which discusses a metaplectic Bohmian formalism from classical (Hamilton's equations) to quantum physics (Schrödinger's equation): the Metatron;

The book is written in a lucid style, nicely marrying physical intuition with mathematical insight. As such, it should be of interest to workers in Schrodinger theory and related areas, and generally, to those who seek a deeper understanding of some of the linear and nonlinear perspectives of the Schrödinger equation.

ACKNOWLEDGMENTS

The Editors wish to thank Nova Science Publishers, for their invitation to participate in the publication of this book. Moreover, we offer heartfelt thanks to all our invited chapter contributors: without your generosity of scholarship and time, amidst already demanding academic schedules, this book would have no substance. In addition, we thank the Physics and English Departments of Western Kentucky University in Bowling Green, KY, for encouragement and material support during the compilation of this work. Finally, our most profound thanks for the love and support of our families, without which we would have not been able participate in this august work. MT21:42.

Valentino A. Simpao
Hunter C. Little
Editors

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Chapter 6

**SELF-SIMILAR AND TRAVELING-WAVE
ANALYSIS OF THE MADELUNG EQUATIONS
OBTAINED FROM THE SCHRÖDINGER
EQUATION**

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Abstract

In 1926, at the advent of quantum mechanics Madelung gave a hydrodynamical foundation of the Schrödinger equation and raised questions which remain unanswered today. In the following chapter, we analyze the multi-dimensional Madelung equations with the self-similar and traveling-wave Ansätze derived from the linear Schrödinger equations (SE) for free particles in Cartesian coordinates. These trial functions are powerful tools to attack various non-linear partial differential equations and find—physically relevant—dispersive or wave-like solutions. The SE in general is a complex partial differential equation. In certain cases, depending on the Hamiltonian, SE resembles the transport equations as the diffusion equation or the Euler equation. We start our study along this

line, arriving from relatively simple to more complex cases. The most interesting results are the shape of the fluid density function, which has a highly oscillatory behavior with an infinite number of zeros which should have a quantum mechanical origin. Such types of density functions are unthinkable for classical fluids. There is a direct connection between the zeros of the Madelung fluid density and the magnitude of the quantum potential as well. In the last part of the study, we present self-similar solutions for the Gross-Pitaevskii or non-linear SE which is rewritten in the Madelung form as well. The obtained system of differential equations is highly non-linear. Regarding the solutions, a larger coefficient of the non-linear term yields stronger deviation of the solution from the linear case. We hope that our present study helps to attract attention to investigate the role of the Madelung equation. Parallel to the well-known Copenhagen quantum mechanical interpretation, there has been a looming and poorly understood hydrodynamical world hidden behind the SE from the very beginning.

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Keywords: partial differential equations, quantum mechanics, fluid dynamics mathematical formulation

1. INTRODUCTION

This book is dedicated to understanding the Schrödinger equation (SE) which is ambitious and energizes the reader with optimism. The market of physics books is overloaded with textbooks which provide an introduction to quantum mechanics, but most are just large cookbooks filled with sophisticated technical information, details, and algorithms on how the theory should be successfully applied to calculate some real numbers as observables (e.g., cross sections) leaving no room for deeper understanding through various interpretations. We do not know how the reader will understand the SE in general after reading this book. First, we try to emphasize the diffusion equation property of the SE showing how the free particle solution can be evaluated with methods which originally came from transport phenomena. Later, we plan to give a “twisted view” of the issues in this chapter and enlighten the hydrodynamical formulation of the SE with some analytic results. One of our results is so new that it is not yet published elsewhere. The hydrodynamical interpretation of the SE is a not-so-well-known viewpoint (usually not part of the regular quantum mechanical university curriculum), but one which is worth learning and understanding.

Our investigation is far from complete, and we are just at the beginning of a long journey. In this chapter, we investigate the solutions of the Madelung form of the linear and free SE; the role of an additional potential, minimal coupling of external magnetic fields, spin, or multi-particle Hamiltonian are outside of our recent scope. We show the reader how to apply the self-similar and traveling-wave trial functions for any kind of Partial Differential Equations (PDE) to obtain reasonable Ordinary Differential Equation (ODE) systems which can be solved with quadratures. Technical parts of the calculation are written in the Appendix. Therefore, we encourage the reader to start and perform such investigations because the track is open.

Beyond the Copenhagen interpretation, there are numerous non-mainstream interpretations that exist for understanding quantum mechanics (without completeness we mention some of them) such as ensemble interpretation by Born [1], quantum logic by Garrett-Birkhoff [2], many word by Ewerett [3] or the de Broglie-Bohm theory [4, 5] which lies closest to our Madelung picture [6, 7]. There are interesting monographs available which summarize numerous interpretations of quantum mechanics like [8, 9, 10]. To be unorthodox, we personally prefer the corresponding Wikipedia page [11] for a quick and clear orientation about all these existing theories. We draw the reader's attention to the work of Bush [12, 13] who enlightens the classical analogue of the de Broglie pilot-wave theory with classical small droplets bouncing across a vibrating liquid bath displaying many features reminiscent of quantum systems. In an interesting way, we have studied a classical model of a particle bouncing on a corrugated surface, which vibrates [14, 15]. A spatially extended transport quantum model one may find in [16, 17].

The structure of the chapter is as follows: first, we give a transport equation solution of the free one-dimensional Cartesian SE to show the connection to the regular diffusion equation. The fact that the SE is a regular diffusion equation with a complex diffusion constant and not a second-order wave equation or a first-order Euler type equation (which is also a wave equation) is rarely voiced.

Then, we explain how the Madelung equation is obtained from the SE, mentioning the de Broglie-Bohm pilot-wave theory and the general properties of the system. In the second half of the chapter, as new features, we derive the multi-dimensional solutions of the free Madelung equations with the self-similar and traveling-wave Ansätze. (The German meaning of trial-function is "Ansatz" and the plural of this word is "Ansätze." We use both expressions in the following.)

2. TRANSPORT PHENOMENA AND THE SCHRÖDINGER EQUATION

One way of thinking is to associate the SE with transport equation, where possible. By doing so, we try to use our experiences related to the description of transport phenomena. The general form

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \quad (1)$$

(where Ψ is the wave function and H is the Hamiltonian of the corresponding particle) can be written for free particle as

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi. \quad (2)$$

For free particle in one dimension we have

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2}. \quad (3)$$

This form can be rearranged in the following way

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2}. \quad (4)$$

This is a standard time evolution equation and resembles to the diffusion equation

$$\frac{\partial \Psi}{\partial t} = D \frac{\partial^2 \Psi}{\partial x^2}, \quad (5)$$

where, the diffusion coefficient is

$$D = \frac{i\hbar}{2m}. \quad (6)$$

In the above presented case, the equation can be viewed as a diffusion equation with a complex diffusion coefficient. Of course this is a mathematical point of view because the diffusion coefficient in the real world is a real number. One may try to use methods which are used in transport phenomena to solve this equation. Looking for solutions in the separable form of the variables of

$$\Psi = A(t)B(x) \quad (7)$$

then we have

$$\frac{\partial A}{\partial t} B = DA \frac{\partial^2 B}{\partial x^2}. \quad (8)$$

By this one obtains

$$\frac{1}{DA} \frac{\partial A}{\partial t} = \frac{1}{B} \frac{\partial^2 B}{\partial x^2}. \quad (9)$$

These two sides are equal for any time and space coordinate, if they are equal to a constant, which we note using $-\epsilon$. From this, we get two equations

$$\frac{1}{DA} \frac{\partial A}{\partial t} = -\epsilon \quad (10)$$

and

$$\frac{1}{B} \frac{\partial^2 B}{\partial x^2} = -\epsilon. \quad (11)$$

This direction of solving the equation is quite well known. Here we mention the reference [18]

$$A = A_0 \exp(-D\epsilon t) \quad (12)$$

and the second has

$$B = B_0 \sin(\sqrt{\epsilon}x). \quad (13)$$

In the second equation, in principle, one may also have a cosine term, but if we constrain ourselves to a free particle in a box, with $\Psi(x=0) = \Psi(x=l) = 0$, then just the sine term counts. Even more, the boundary conditions yield

$$B = B_0 \sin\left(\frac{n\pi}{l}x\right) \quad (14)$$

where the relation

$$\sqrt{\epsilon} = \frac{n\pi}{l} \quad (15)$$

was used, with $n = 1, 2, 3, \dots$. Inserting the above values for ϵ , the solution for A becomes

$$A = A_0 \exp\left(-D \frac{n^2 \pi^2}{l^2} t\right). \quad (16)$$

Consequently, the result is

$$\Psi = \Psi_{0,n} \exp\left(-D \frac{n^2 \pi^2}{l^2} t\right) \cdot \sin\left(\frac{n\pi}{l}x\right), \quad (17)$$

which satisfies the equation under the conditions presented above. In a strict sense, we have a solution for each n . At this point, we reinsert our complex “diffusion coefficient,” and we get

$$\Psi = \Psi_{0,n} \exp\left(-\frac{i\hbar n^2\pi^2}{2m l^2}t\right) \cdot \sin\left(\frac{n\pi}{l}x\right). \quad (18)$$

If we use the Euler formula, then we obtain, for the time-dependent solution,

$$\Psi = \Psi_{0,n} \left[\cos\left(-\frac{\hbar n^2\pi^2}{2m l^2}t\right) - i \sin\left(-\frac{\hbar n^2\pi^2}{2m l^2}t\right) \right] \cdot \sin\left(\frac{n\pi}{l}x\right). \quad (19)$$

It is now clear that the solutions are pure plane waves. The probability density of the system reads

$$\rho_{prob} = |\Psi^* \Psi|^2 = \Psi_{0,n}^2 \sin^2\left(\frac{n\pi}{l}x\right). \quad (20)$$

We may try to solve the free SE with the self-similar and traveling-wave Ansatz (the detailed analysis of these trial functions will come later).

3. THE MADELUNG EQUATION

Finding classical physical basements of quantum mechanics is a great challenge since the advent of the theory. Madelung was one among the firsts who gave one explanation; this was the hydrodynamical foundation of the SE [6, 7]. His exponential transformation simply indicates that one can model quantum statistics hydrodynamically. Later, it became clear that the Madelung Ansatz is just the complex Cole-Hopf transformation [19, 20] which is sometimes used to linearize non-linear partial differential equations (PDEs). A classical example is the Kardar-Parisi-Zhang (KPZ) dynamic interface growth equation which is a regular diffusion equation containing a non-linear gradient square term, which can be eliminated with the Hopf-Cole transformation. In our recent study [21], we present numerous analytic solutions for the KPZ equation, and each solution shows the fingerprint of this transformation.

Following the original paper of Madelung [7], the time-dependent SE reads

$$\Delta\Psi - \frac{8\pi^2m}{\hbar^2}U\Psi - i\frac{4\pi m}{\hbar}\frac{\partial\Psi}{\partial t} = 0, \quad (21)$$

where Ψ , U , m and h are the wave function, potential, mass, and Planck's constant, respectively.

Let's take the following Ansatz of Madelung $\Psi = \sqrt{\rho}e^{iS}$ where $\rho(x, t)$ and $S(x, t)$ are continuous time- and space-dependent functions. (Both $\rho(x, t)$ and $S(x, t)$ should have existing continuous first time and second spatial derivatives.) The square root of the amplitude of the original wave function is considered as fluid density ρ and the gradient of the wave function phase $\mathbf{v} = (\hbar/m)\nabla S$ is the velocity field.

Substituting this trial function into (21), the real and the complex part give us the following continuity and Euler equations with the form of

$$\begin{aligned}\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0, \\ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} &= \frac{h^2}{8\pi^2 m^2} \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{1}{m} \nabla U.\end{aligned}\quad (22)$$

Madelung also showed that this is a rotation-free flow. With this unusual transformation, we decompose the second-order diffusion equation into two first-order Euler equations. (Note, that a general Euler equation is a first-order wave equation $u_x + c \cdot u_t = 0$ with a traveling-wave solution of $u = f(x - \frac{t}{c})$ traveling to the right on the axis with finite real-wave velocity of c .) These are relevant and rarely made statements. The physical meaning might be quite general such that diffusion phenomena could be transformed to two coupled first-order wave phenomena. Of course, we cannot give a rigorous mathematical proof for this statement, but it helps our physical intuitive picture of understanding. Some kind of reverse picture is also true—and more commonly known in physics—considering the Fourier or Fick law with the continuity equation (which are first-order PDEs) gives us the heat conduction or diffusion equation (the Fourier law has to be substituted into the continuity equation to the heat conduction equation).

The transformed equations have an attractive feature: the Planck's constant appears only once, at the coefficient of the quantum potential or pressure, which is the first term of the right-hand side of the second equation. Note, that these are most general vectorial equations for the velocity field v which means that one-, two-, or three-dimensional motions can be investigated as well. In the following, we will consider the two-dimensional flow motion $\mathbf{v} = (u, v)$ in Cartesian coordinates without any external field $U = 0$. The functional form of the three- and one-dimensional solutions will be mentioned briefly as well.

It is important to mention, that in the '50s and '60s of the twentieth century Jánossy [22, 23] derived the Madelung form of various, more complex SE, including external magnetic field with the minimal coupling, electron spin, or even for multi-particle systems.

The quantum potential also appears in the de Broglie-Bohm pilot-wave theory [4, 5] (in other contexts) which is a non-mainstream attempt to interpret quantum mechanics as a deterministic non-local theory. In the case of $\hbar \rightarrow 0$, the Euler equation goes over the Hamilton-Jacobi equation.

Another consistency check of the Madelung Ansatz is the following. What happens when we apply it to the time-independent SE? (We show it for one dimension only.) Let's try!

$$-\frac{\hbar^2}{2m}\Delta\Psi(x) + V(x)\Psi(x) = E\Psi(x) \quad (23)$$

Here, the Ansatz is changed to a simpler form of $\Psi(x) = A(x)e^{-iB(x)}$. After some trivial algebraic manipulation, we obtain the coupled ODE system of

$$\begin{aligned} -\frac{\hbar^2}{2m}A'' + A[B']^2 + VA - EA &= 0, \\ -\frac{\hbar^2}{m}A'B' - \frac{\hbar^2}{2m}AB'' &= 0, \end{aligned} \quad (24)$$

where prime means derivation with respect to the spatial coordinate x . With the substitution of $B(x) = \text{const.}$, we get back the usual SE. Fixing the interaction potential to the usual Coulomb, quadratic, or $\text{Sin}(x)$ functions, we obtain the regular Coulomb, harmonic oscillator, or Mathieu [24] function solutions. The $B(x) \neq \text{const.}$ choice leads to formulas containing non-analytic formulas with formal integration of special functions. In-depth analysis of this problem might not be useless. A similar, parallel derivation can be done for purely time-dependent SEs as well:

$$i\hbar\frac{\partial\Psi(t)}{\partial t} = V\Psi(t). \quad (25)$$

Using the proper Ansatz of $\Psi(t) = A(t)e^{iB(t)}$, we obtain the next two equations for the real and imaginary parts:

$$\begin{aligned} A\dot{B} &= AV, \\ \dot{A} &= 0, \end{aligned} \quad (26)$$

where dot means derivation with respect to time. For the harmonic potential $V = \cos(\omega t)$, the obtained ODE can be easily integrated, resulting

$$B(t) = \frac{\sin(\omega t)}{\omega} + c_1. \quad (27)$$

We investigate the mathematical properties of the Madelung hydrodynamical equation but, for clarity, we briefly explain how the Madelung equation is related to the de Broglie or Bohm pilot-wave theory.

Both theories describe the wave-particle duality in a common manner. Each states that the usual SE describes the pilot-wave (now a wave motion in a media which propagated around a point-like particle) and an additional dynamical equation. Such an additional dynamical equation describes the velocity or the acceleration of the point particle evaluated from the phase of the wave function or from a Newton equation where the gradient of quantum potential is added to the classical interaction. These equations are the following

$$\mathbf{v} = (\hbar/m)\nabla S, \quad (28)$$

and

$$m \frac{d^2 r}{dt^2} = -\nabla \left[V + \frac{\hbar^2}{8\pi^2 m} \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) \right]. \quad (29)$$

This explanation can be found in the monograph of Peña [25] or in the work of Dewdney *et al.* [26].

As a peculiarity, Wallstrom showed, with mathematical means, that the initial-value problem of the Madelung equation is not well-defined and additional conditions are needed [27].

Nowadays, hydrodynamical description of quantum mechanical systems is a popular technical tool in numerical simulations. Review articles on quantum trajectories can be found in a booklet edited by Hughes in 2011 [28].

From general concepts, as the second law of thermodynamics, a weakly non-local extension of ideal fluid dynamics can be derived which leads to the Schrödinger-Madelung equation as well [29].

To our knowledge, there are no direct analytic solutions available for the Madelung equation. Baumann and Nonnenmacher [30] exhaustively investigated the Madelung equation with Lie transformations and presented numerous ODEs; however, no exact and explicit solutions are presented in a transparent

way. Additional studies exist where the non-linear Schrödinger equation is investigated with the Madelung Ansatz ending up with solitary wave solutions [31]; however, that is not the field of our present interest.

The application of the original Madelung (or Cole-Hopf) transformation grew out of the linear quantum theory and was applied to more sophisticated field theoretical problems. Recently, Chavanis and Matos [32] applied the Madelung Ansatz to the Klein-Gordon-Maxwell-Einstein equation in curved space-time, possibly describing self-gravitating Bose-Einstein condensates, coupled to an electromagnetic field. This study clearly presents the wide applicability of the original Madelung Ansatz to refined quantum field theories with the possible hydrodynamic interpretation picture.

4. SELF-SIMILAR ANALYSIS

There are no existing methods which could help us to solve PDE in general. However, two basic linear and time-dependent PDE exist which could help us.

The first is the most important linear PDE which is the diffusion or heat conduction equation. (The second one is the wave equation and will be analyzed later.) In one dimension it reads

$$\frac{\partial u(x, t)}{\partial t} = a \frac{\partial^2 u(x, t)}{\partial t^2}, \quad (30)$$

where a is the diffusion or heat conduction coefficient and should be a positive real number. The well-known solution is the Gaussian curve which describes the decay and spreading of the initial particle or heat distribution. There is a rigorous mathematical theorem “the strong maximum principle” which states that the solution of the diffusion equation is bounded from above. The problem of this equation is the infinite signal propagation speed, which basically means that the Gaussian curve has no compact support, but that is not a relevant topic now. The main point “which is the major motivation of this chapter” is that there exists a natural Ansatz (or trial function) which solves this equation. This is the self-similar solution, the corresponding one-dimensional mathematical form is well-known from various textbooks [33, 34, 35]

$$T(x, t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right) := t^{-\alpha} f(\eta), \quad (31)$$

where $T(x, t)$ can be an arbitrary variable of a PDE and t means time and x means spatial dependence. The similarity exponents α and β are of primary

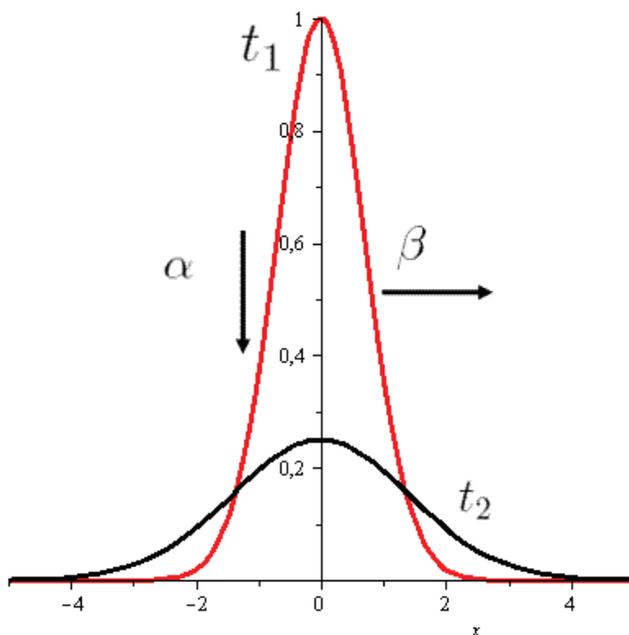


Figure 1. A self-similar solution of (31) for $t_1 < t_2$. The presented curves are Gaussians for regular heat conduction.

physical importance since α gives the rate of decay of the magnitude $T(x, t)$, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution as time goes on. The most powerful result of this Ansatz is the fundamental or Gaussian solution of the Fourier heat conduction equation (or for Fick diffusion equation) with $\alpha = \beta = 1/2$.

The equation above can be rewritten

$$\frac{\partial u}{\partial(at)} = \frac{\partial^2 u}{\partial x^2}. \quad (32)$$

With the change of variable

$$\tau = at \quad (33)$$

we get

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \quad (34)$$

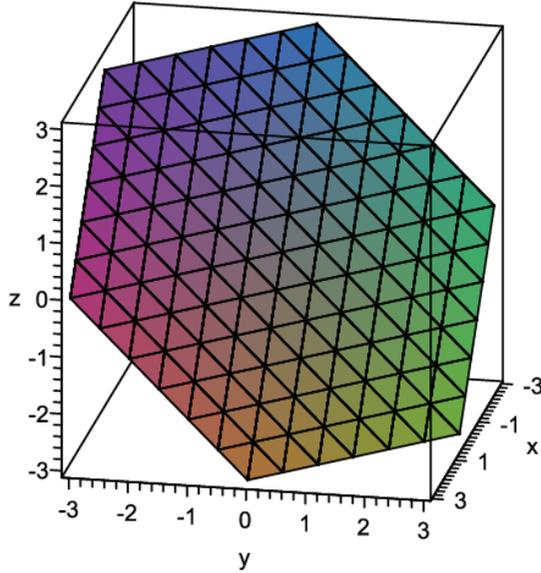


Figure 2. The graph of the $x + y + z = 0$ plane.

As previously mentioned, by appropriate scaling, we can try to reduce the PDE to an ODE. Using the values of α and β presented, we arrive, for the function u , at the following form,

$$u = B \frac{1}{\tau^{1/2}} f \left(\frac{x}{\tau^{1/2}} \right). \quad (35)$$

Now we introduce the variable

$$\eta = \frac{x}{\tau^{1/2}}. \quad (36)$$

We evaluate the terms in the equation. For convenience, we choose for $B = 1$: the constant can be reintroduced later. The equation related to the self-similar transformations is also mentioned by Barenblatt [34] which we will not discuss in more detail. The time derivative is now

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \tau} \left(\tau^{1/2} f(\eta) \right). \quad (37)$$

Carrying out the evaluation, we get

$$\frac{\partial u}{\partial \tau} = -\frac{1}{2}\tau^{-\frac{3}{2}}f(\eta) - \frac{1}{2}\tau^{-\frac{3}{2}}\frac{x}{\tau^{1/2}}\frac{\partial f}{\partial \eta}. \quad (38)$$

For the derivative in the space variable, we have

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\tau}}\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{\tau}}\frac{\partial f}{\partial \eta}\frac{1}{\sqrt{\tau}}. \quad (39)$$

For the second derivative in the space variable

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\tau}\frac{\partial^2 f}{\partial \eta^2}\frac{\partial \eta}{\partial x} = \tau^{-\frac{3}{2}}\frac{\partial^2 f}{\partial \eta^2}. \quad (40)$$

If we insert these expressions in the differential equation, we then obtain the equation for f with certain variables

$$-\frac{1}{2}\tau^{-\frac{3}{2}}f(\eta) - \frac{1}{2}\tau^{-\frac{3}{2}}\frac{x}{\tau^{1/2}}\frac{\partial f}{\partial \eta} = \tau^{-\frac{3}{2}}\frac{\partial^2 f}{\partial \eta^2}. \quad (41)$$

This equation should be valid for any positive τ , and taking into account that $x/\sqrt{\tau} = \eta$, we have the following ODE

$$2\frac{\partial^2 f}{\partial \eta^2} + \eta\frac{\partial f}{\partial \eta} + f = 0. \quad (42)$$

This equation has the solution

$$f(\eta) = e^{-\frac{\eta^2}{4}}. \quad (43)$$

Form this we get, for the function u

$$u = \frac{B}{\sqrt{\tau}}e^{-\frac{\eta^2}{4}} = \frac{B}{\sqrt{\tau}}e^{-\frac{x^2}{4\tau}}. \quad (44)$$

If we reinsert the value $\tau = at$, then we have

$$u = \frac{B}{\sqrt{at}}e^{-\frac{\eta^2}{4}} = \frac{B}{\sqrt{at}}e^{-\frac{x^2}{4at}}. \quad (45)$$

This is a relative general form of u , which solves the equation presented above and now we try to see the consequences regarding the SE. The a stands

for a kind of diffusion coefficient in the equation of u . If we use the idea at the beginning of this study, and we insert, instead of a , the ratio $\frac{i\hbar}{2m}$, then we obtain

$$u = \frac{B}{\sqrt{\frac{i\hbar}{2m}}} e^{-\frac{2x^2m}{4i\hbar t}}. \quad (46)$$

The exponent can be split into real and imaginary parts part using Euler formula

$$u = B\sqrt{\frac{2m}{i\hbar}} \cos\left(\frac{x^2 2m}{4\hbar t}\right) + iB\sqrt{\frac{2m}{i\hbar}} \sin\left(\frac{x^2 2m}{4\hbar t}\right). \quad (47)$$

This is already a form which corresponds to the equation with complex coefficient. The form at which we arrived can be brought closer to interpretation. Expressing the \sqrt{i} in the denominator, we have

$$u = \sqrt{\frac{2m}{\hbar}} B \frac{1}{\sqrt{t}} \frac{\cos\left(\frac{x^2 2m}{4\hbar t}\right) + i \sin\left(\frac{x^2 2m}{4\hbar t}\right)}{\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}}, \quad (48)$$

and

$$u = \sqrt{\frac{2m}{\hbar}} B \frac{1}{\sqrt{t}} \frac{\cos\left(\frac{x^2 2m}{4\hbar t}\right) + i \sin\left(\frac{x^2 2m}{4\hbar t}\right)}{-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}}. \quad (49)$$

One may simplify the expression above as

$$u = B\sqrt{\frac{m}{\hbar}} (1 - i) \frac{\cos\left(\frac{x^2 m}{2\hbar t}\right) + i \sin\left(\frac{x^2 m}{2\hbar t}\right)}{\sqrt{t}} \quad (50)$$

and

$$u = B\sqrt{\frac{m}{\hbar}} (-1 + i) \frac{\cos\left(\frac{x^2 m}{2\hbar t}\right) + i \sin\left(\frac{x^2 m}{2\hbar t}\right)}{\sqrt{t}}. \quad (51)$$

By this we arrive to a complex form

$$u = \pm \sqrt{m\hbar} \frac{1}{\sqrt{t}} \left[\cos\left(\frac{x^2 m}{2\hbar t}\right) + \sin\left(\frac{x^2 m}{2\hbar t}\right) + i \left(\sin\left(\frac{x^2 m}{2\hbar t}\right) - \cos\left(\frac{x^2 m}{2\hbar t}\right) \right) \right]. \quad (52)$$

These solutions are visualized on Fig. 2 for different time-points $t_1 < t_2$. Applicability of this Ansatz is quite wide and comes up in various transport

systems [33, 34, 35, 36, 37, 38]. Solutions with integer exponents are called self-similar solutions of the first kind (and sometimes can be obtained from dimensional analysis of the problem as well). The above given Ansatz can be generalized considering real and continuous functions $a(t)$ and $b(t)$ instead of $t^{-\alpha}$ and t^β having the form of $a(t)f\left(\frac{x}{b(t)}\right)$. At this point, we must mention the general similarity solutions of Bluman and Cole [39] who exhaustively studied the heat conduction equation.

Our self-similar transformation is based on the assumption that a self-similar solution exists; for example, every physical parameter preserves its shape during the expansion. Self-similar solutions usually describe the asymptotic behavior of an unbounded or a far-field problem. The time t and the space coordinate x appear only in the combination of $f(x/t^\beta)$. It means that the existence of self-similar variables implies the lack of characteristic length and time scales. These solutions are usually not unique and do not take into account the initial stage of the physical expansion process. These kind of solutions describe the intermediate asymptotic of a problem: they hold when the precise initial conditions are no longer important, but before the system has reached its final steady state. For some systems it can be shown that the self-similar solution fulfills the source type (Dirac delta) initial condition, but not in every case. These Ansätze are much simpler than the full solutions of the PDE and thus, are easier to understand and study in different regions of parameter space. A final reason for studying them is that they are solutions of a system of ODE and hence, do not suffer the extra inherent numerical problems of the full PDE. In some cases, self-similar solutions help to understand diffusion-like properties or the existence of compact supports of the solution.

Within the introduction of this method, an important comment has to be made. The functional form (53) is perfect from the mathematical point of view, but incorrect if it is applied to a physical system where the variables should have well-defined physical dimensions. Our present system—the Madelung equation—is a nice example. The dynamical variables are the fluid density (with dimension of $[kg/m^3]$) and fluid velocity (with dimension of $[m/s]$), if we use SI units. So the original form of

$$T(x, t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right) := t^{-\alpha} f(\eta), \quad (53)$$

has to be modified at two points:

(1) the argument of the shape-function $[\eta]$ should be dimensionless and

(2) the complete form of the solution $t^{-\alpha} f(\eta)$ should have a proper dimension (e.g., kg/m^3 or m/s). This problem can be easily solved if we introduce two additional constants (like a and b) with unit numerical values which fix the proper dimension. So

$$T(x, t) = at^{-\alpha} f\left(\frac{x}{bt^\beta}\right) := at^{-\alpha} f(\eta/b), \quad (54)$$

if T stands for the density with $[kg/m^3]$ SI dimension and $\alpha = 1$ then the dimension of a is $[\frac{kg \cdot s}{m^3}]$ the numerical value is unity. Additionally, if $\beta = 1/2$ then the dimension of b is $[\frac{m}{s^2}]$. Other exponents like $\beta = 3/2$ should dictate other dimensions for the constants. All these constants have unit numerical values, therefore, we neglect them in all of our studies for a better transparency. This attitude is similar to the application of nuclear physics units in quantum mechanics.

This Ansatz can be generalized for two or three dimensions in various ways, one of which is the following

$$u(x, y, z, t) = t^{-\alpha} f\left(\frac{F(x, y, z)}{t^\beta}\right) := t^{-\alpha} f\left(\frac{x + y + z}{t^\beta}\right) := t^{-\alpha} f(\omega) \quad (55)$$

where $F(x, y, z)$ can be understood as an implicit parameterization of a two-dimensional surface. If $F(x, y, z) = x + y + z = 0$ which is presented on Fig. 3 then it is an implicit form of a plane in three-dimensions. Now we can give a geometrical interpretation of the Ansatz. Note that the dimension of $F(x, y, z)$ still has to be a spatial coordinate. With this Ansatz we consider all $u(x, y, z, t = t_0)$ functions where the sum of the spatial coordinates are on a plane $F(x, y, z) = x + y + z = 0$ on the same footing. We are not considering the R^3 , u field, but only one plane of the u as an independent variable. The corresponding PDE—which is responsible for the dynamics (e.g., a heat conduction equation)—maps this kind of u functions (this plane now) which are on the surface of another geometry. In this sense, we can investigate the dynamical properties of the equation truly. For more details see [37]. In principle, there are more possible generalizations of the Ansatz available. One is the following:

$$u(x, y, z, t) = t^{-\alpha} f\left(\frac{\sqrt{x^2 + y^2 + z^2 - a}}{t^\beta}\right) := t^{-\alpha} f(\omega) \quad (56)$$

which can be interpreted as an Euclidean vector norm or L^2 norm. Now we contract all the u (which are on a surface of a sphere with radius a) to a simple spatial coordinate. Unfortunately, if we consider the first and second spatial derivatives and plug them into a Euler type equation (like the Madelung equation) we cannot get a pure ω dependent ODE system some explicit x, y, z or t dependence tenaciously remain. For a telegraph-type heat conduction equation (where is no $\mathbf{v}\nabla\mathbf{v}$) term, both of these Ansätze are useful to get solutions for the two-dimensional case [40].

Finally, it is important to emphasize that the self-similar Ansatz has a relevant, but not well-known and not rigorous connection to phase transitions and critical phenomena. Namely to scaling, universality, and renormalization. As far as we know, even genuine pioneers of critical phenomena like Stanley [41] cannot provide rigorous clear-cut definitions for all these conceptions, but we feel that all have a common root. The starting point could be the generalized homogeneous function like the Gibbs potential $G_s(H, \epsilon)$ for a spin system. Close to the critical point, the scaling hypothesis can be expressed via the following mathematical rule $G_s(\lambda^a H, \lambda^b \epsilon) = \lambda G_s(H, \epsilon)$. Where H is the order parameter, the magnetic field and ϵ is the reduced temperature, a, b are the critical exponents. The same exponents mean the same universality classes. The equation gives the definition of homogeneous functions. Empirically, one finds that all systems in nature belong to one of a comparatively small number of such universality classes. The scaling hypothesis predicts that all the curves of this family $M(H, \epsilon)$ can be "collapse" onto a single curve provided one plots not M versus ϵ but rather a scaled M (M divided by H to some power) versus a scaled ϵ (ϵ divided by H to some different power). The renormalization approach to critical phenomena leads to scaling. In renormalization, the exponent is called the scaling exponent. We hope that this small turn-out allows the reader to gain a better understanding of our approach.

In past years, we successfully applied the multi-dimensional generalization of the self-similar Ansatz to numerous viscous fluid equations [38, 42] ending up with a book chapter of [43].

Without the loss of generality, we investigate the two-dimensional Madelung equation with the following Ansatz

$$\rho(x, y, t) = t^{-\alpha} f\left(\frac{x+y}{t^\beta}\right) := t^{-\alpha} f(\eta), \quad u(x, y, t) = t^{-\delta} g(\eta), \quad v(x, y, t) = t^{-\epsilon} h(\eta), \tag{57}$$

where $f, g,$ and h are the shape functions of the density and the velocity field, respectively. The similarity exponents $\alpha, \beta, \delta, \epsilon$ are of primary physical importance since α, δ, ϵ represent the damping of the magnitude of the shape function while β represents the spreading. With the exception of some pathological cases, all positive similarity exponents mean physically relevant dispersive solutions with decaying features at $x, y, t \rightarrow \infty$. Substituting the Ansatz (57) into (22) and performing some algebraic manipulation (which is elaborated in the Appendix), the next ODE system can be expressed for the shape functions

$$\begin{aligned} -\frac{1}{2}f - \frac{1}{2}f'\eta + f'g + fg' + f'h + fh' &= 0, \\ -\frac{1}{2}g - \frac{1}{2}g'\eta + gg' + hg' - \frac{\hbar^2}{2m^2} \left(\frac{f'^3}{2f^3} - \frac{f'f''}{f^2} + \frac{f'''}{2f} \right) &= 0, \\ -\frac{1}{2}h - \frac{1}{2}h'\eta + gh' + hh' - \frac{\hbar^2}{2m^2} \left(\frac{f'^3}{2f^3} - \frac{f'f''}{f^2} + \frac{f'''}{2f} \right) &= 0. \end{aligned} \tag{58}$$

The first continuity equation can be integrated giving us the mass as a conserved quantity and the parallel solution for the velocity fields $\eta = 2(g+h) + c_0$ where c_0 is the usual integration constant, which we set to zero. (A non-zero c_0 remains an additive constant in the final ODE (59) as well.) It is interesting and unusual (in our practice) that even the Euler equation can be integrated once giving us the conservation of momenta. For classical fluids this is not the case. After some additional algebraic steps, a decoupled ODE can be derived for the shape function of the density

$$2f''f - (f')^2 + \frac{m^2\eta^2f^2}{2\hbar^2} = 0. \tag{59}$$

All the similarity exponents have the fixed value of $+\frac{1}{2}$ which is usual for regular heat conduction, diffusion, or for Navier-Stokes equations [43]. Note, that the two remaining free parameters are the mass of the particle m and \hbar which is the Planck's constant divided by 2π . For a better transparency, we fix $\hbar = 1$. This is consistent with the experience of regular quantum mechanics that quantum features are relevant at small particle masses.

At this point, it is worth mentioning that the obtained ODE for the density shape function is very similar to (59) for different space dimensions, the only difference is a constant in the last term. For one, two, or three dimensions, the denominator has a factor of 1, 2, or 3, respectively.

An additional space dependent potential U (like a dipol, or harmonic oscillator interaction) in the original Schrödinger equation would generate an extra fifth term in (59) like $\eta, f(\eta), \eta^2$. Unfortunately, no other analytic closed form solutions can be found for such terms.

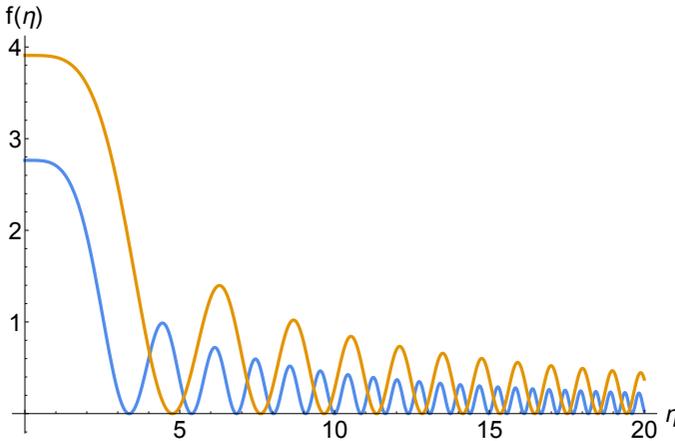


Figure 3. The solution of (60) ($c_1 = c_2 = 1$) the red and dashed curve is for $m = 1$ and the black and solid curve is for $m = 0.5$.

The solution of (59) can be expressed with the help of the Bessel functions of the first and second kind [44] and has the following form of

$$f(\eta) = \frac{2 \left(-J_{\frac{1}{4}} \left[\frac{\sqrt{2}m\eta^2}{8} \right] \cdot c_1 + Y_{\frac{1}{4}} \left[\frac{\sqrt{2}m\eta^2}{8} \right] \cdot c_2 \right)^2}{\eta^3 m^2 \left(J_{-\frac{3}{4}} \left[\frac{\sqrt{2}m\eta^2}{8} \right] \cdot Y_{\frac{1}{4}} \left[\frac{\sqrt{2}m\eta^2}{8} \right] - J_{\frac{1}{4}} \left[\frac{\sqrt{2}m\eta^2}{8} \right] \cdot Y_{-\frac{3}{4}} \left[\frac{\sqrt{2}m\eta^2}{8} \right] \right)^2} \tag{60}$$

where c_1 and c_2 are the usual integration constants. The correctness of this solutions can be easily verified via back substitution into the original ODE.

To imagine the complexity of these solutions, Fig. 3 and Fig. 4 present $f(\eta)$ for various m, c_1 and c_2 values. It has a strong decay with a stronger and stronger oscillation at large arguments. The function is positive for all values of

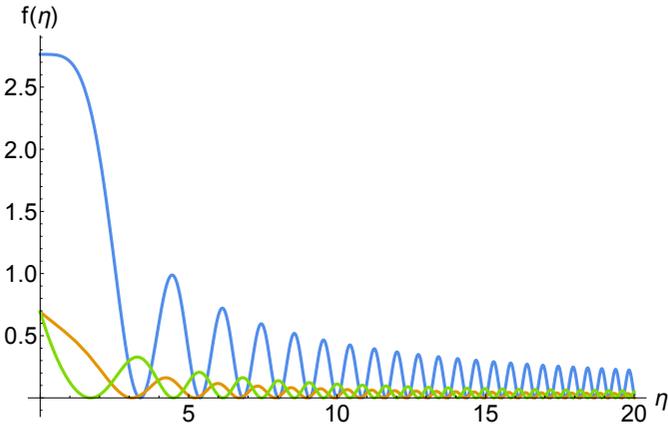


Figure 4. The solution of (60) ($m = 1$) the blue curve is for $c_1 = c_2 = 1$, the yellow for $c_1 = 1/4$ and $c_2 = 1/2$, and the green for $c_1 = -1/2$ and $c_2 = 1/2$.

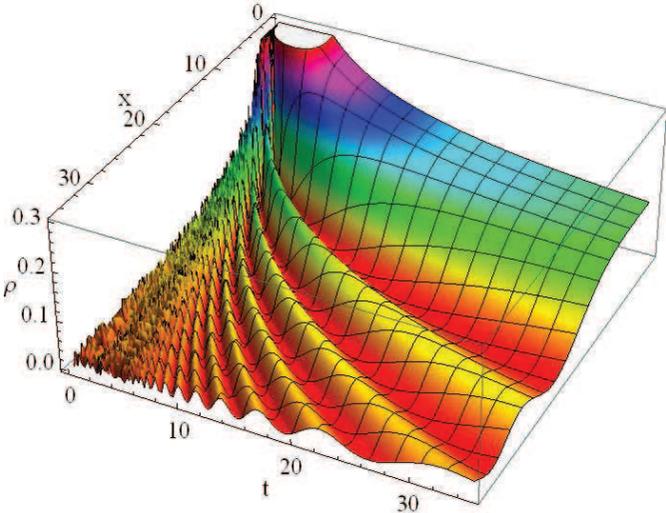


Figure 5. The density function $\rho(x, y = 1, t)$ with the $c_1 = c_2 = m = 1$ parameter set.

the argument (which is physical for a fluid density), but such oscillatory profiles are completely unknown in regular fluid mechanics [43]. The most interesting feature is the infinite number of zero values which cannot be interpreted physically for a classical real fluid.

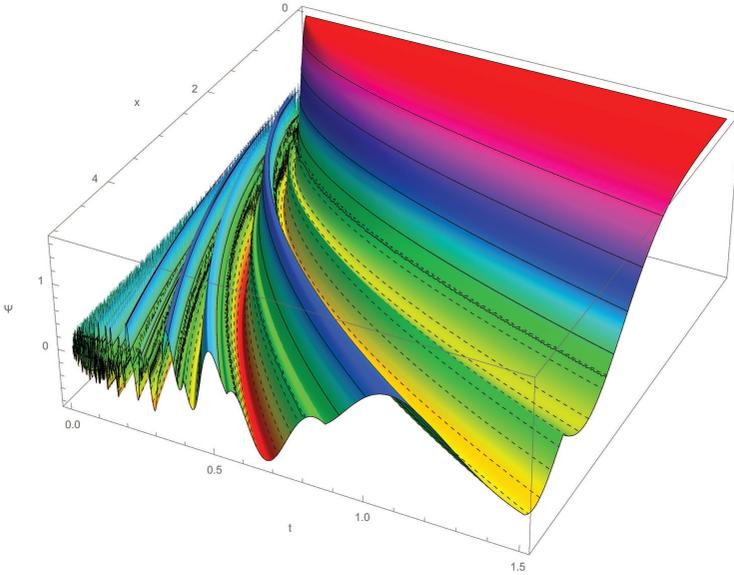


Figure 6. The projection of the real part of the wave function $\Psi(x, t)$ from (63) for $m = 1 = c_1 = c_2$.

The presented form of the shape function cannot be simplified further, only Y_ν s can be expressed with the help of J_ν s [44]. Applying the recurrence formulas, the orders of the Bessel functions can be shifted as well. With the parabolic cylinder functions the Bessel functions with $3/4$ and $1/4$ orders can be expressed, too. The denominator of (60) can be simplified to a power function, therefore $f(\eta)$ can be written in a much simpler form:

$$f(\eta) = \frac{\pi\eta}{64} \left(c_1 J_{\frac{1}{4}} \left[\frac{\sqrt{2}m\eta^2}{8} \right] - c_2 Y_{\frac{1}{4}} \left[\frac{\sqrt{2}m\eta^2}{8} \right] \right)^2. \quad (61)$$

Fig. 5 shows the complete density function $\rho(x, y = 1, t)$, note the singularity at the origin.

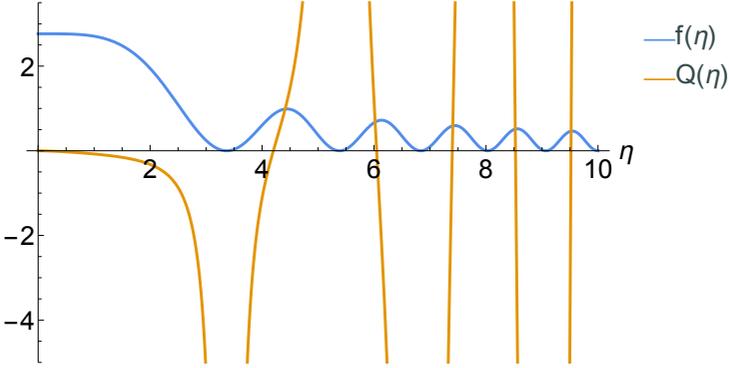


Figure 7. The shape function of the density $f(\eta)$ is the blue curve and the shape function of the quantum potential $Q(\eta)$ is the yellow solid line. All the corresponding parameters are $c_1 = c_2 = m = 1$.

Both J_ν and Y_ν Bessel functions with linear argument form an orthonormal set, therefore integrable over the L^2 space. In our case, the integral $\int_0^\infty f(\eta)d\eta$ is logarithmically divergent; unfortunately, it cannot be interpreted as a physical wave function of the original Schrödinger equation. We have the feeling that the quadratic argument of the shape function of the density might have some physical interpretation (e.g., in Fresnel diffraction, or in Fresnel transformation, the square of the argument is used).

However, \sqrt{f} could be interpreted as the fluid mechanical analogue of the real part of the wave function of the free quantum mechanical particle which can be described with a Gaussian wave packet. To obtain the complete original wave function, the imaginary part has to be evaluated as well. It is trivial from $\eta = \frac{x+y}{t^{1/2}} = 2(g+h)$ that

$$S = \frac{m}{\hbar} \int_{\mathbf{r}_0}^{\mathbf{r}^1} \mathbf{v}d\mathbf{r} = \frac{m}{\hbar} \frac{(x+y)^2}{4t}. \tag{62}$$

Now

$$\Psi(x, y, t) = \frac{\sqrt{\pi}}{8} \frac{(x+y)^{1/2}}{t^{1/4}} \left(c_1 J_{1/4} \frac{\sqrt{2}m(x+y)^2}{8t} - c_2 Y_{1/4} \frac{\sqrt{2}m(x+y)^2}{8t} \right) e^{\frac{im}{\hbar} \frac{(x+y)^2}{4t}}. \tag{63}$$

Fig. 6 shows the projection of the real part wave function to the x, t sub-space. At small times, the oscillations are clear to see; however, at larger times the strong damping is evident.

For arbitrary quantum systems, the wave function can be evaluated according to the SE. However, we never know directly how large the quantum contribution is in comparison to the classical one. Now, it is possible for a free particle to get this contribution. (The Schrödinger equation gives the Gaussian wave function for a freely propagating particle.) With the Madelung Ansatz we obtained the classical fluid dynamical analogue of the motion with the physical parameters $\rho(x, y, t)$, $\mathbf{v}(x, y, t)$ which can be calculated analytically via the self-similar Ansatz, thereafter, the original wave function $\Psi(x, y, t)$ of the quantum problem can be evaluated as well. The magnitude of the quantum potential Q directly informs us where quantum effects are relevant. This can be evaluated from the classical density of the Madelung equation (22) via

$$Q = \frac{\hbar^2}{8\pi^2 m^2} \nabla \cdot \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = \frac{\hbar^2}{8\pi^2 m^2} \frac{\partial}{\partial \eta} \left(\frac{-\eta^2 m^2}{c_1 8J_{\frac{1}{4}} \left[\frac{m\eta^2}{4\sqrt{2}} \right] - c_2 8Y_{\frac{1}{4}} \left[\frac{m\eta^2}{4\sqrt{2}} \right]} \right). \tag{64}$$

Fig. 7 shows the shape function of quantum potential $Q(\eta)$ comparing to the shape function of the density $f(\eta)$. Note, that where the density has zeros, the quantum potential is singular. Such singular potentials might appear in quantum mechanics; however, the corresponding wave function should compensate the effect. This question is analyzed in the book of Holland [46] for various other quantum systems.

5. TRAVELING-WAVE ANALYSIS

The second linear PDE is the hyperbolic second-order wave equation; in one dimension the well-known form is

$$\frac{\partial^2 u(x, t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u(x, t)}{\partial t^2} = 0 \tag{65}$$

where c is the wave propagation velocity (always a finite value), and u is the physical quantity which propagates. Traveling waves, in both directions, are the general solutions of this problem with the form of

$$u(x, t) = f(x \pm ct) = f(\omega). \tag{66}$$

This is a, more-or-less, common knowledge in the theoretical physics community. Gilding and Kersner used the traveling-wave Ansatz to study numerous reaction-diffusion equation systems [47]. To describe pattern formation phenomena [48] the traveling-waves Ansatz is a useful tool as well. Saarloos investigated front propagation into unstable states [49] where traveling-waves play a key role.

To fulfill a complete program of your analysis, we first derive the traveling-wave solution of the one-dimensional SE and later the Madelung system. The former derivation is almost trivial. Let's start with the free SE of

$$\frac{4i\pi m}{\hbar} \frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial x^2} \quad (67)$$

the Ansatz is

$$\Psi(x, t) = f(x - c \cdot t) = f(\eta). \quad (68)$$

By calculating the first time and second spatial derivative of [68] and writing back to [67] we have the linear second-order ODE of the form of

$$-f'' + \frac{4i\pi m}{\hbar} c f' = 0. \quad (69)$$

The solutions are the usual plane waves of

$$f = c_1 + c_2 e^{\frac{4i\pi m}{\hbar} c(x-ct)} \quad (70)$$

with the two integral constants of c_1 , c_2 and wave velocity of c .

This simple trial function can be generalized in numerous ways, for example, to $e^{-\alpha t} f(x \pm ct) := e^{-\alpha t} f(\omega)$ which describes exponential decay or to $g(t) \cdot f(x \pm c \cdot t) := g(t)f(\omega)$ which can even be a power law function of time as well. In 2006, He and Wu developed the so-called exp-function method [50] which relies on an Ansatz (a rational combination of exponential functions), involving many unknown parameters to be specified at the stage of solving the problem. The method soon drew the attention of many researchers. Later, Aslan and Marinakis [51] summarized the various applications of the Ansatz.

There is another existing remarkable Ansatz which interpolates the traveling-wave and the self-similar Ansatz by Benhamidouche [52].

We use the trivial multi-dimensional generalization of the one-dimensional traveling-wave Ansatz of

$$\rho(x, y, t) = f(x + y + ct) = f(\eta), \quad u(x, y, t) = g(\eta), \quad v(x, y, t) = h(\eta). \quad (71)$$

Applying this Ansatz to the Madelung PDE (22) we end up with the ODE system of

$$\begin{aligned} cf' + f'g + fg' + f'h + fh' &= 0, \\ cg' + gg' + hg' - \frac{\hbar^2}{2m^2} \left(\frac{f'^3}{2f^3} + \frac{f'f''}{f^2} + \frac{f'''}{2f} \right) &= 0, \\ ch' + gh' + hh' - \frac{\hbar^2}{2m^2} \left(\frac{f'^3}{2f^3} - \frac{f'f''}{f^2} + \frac{f'''}{2f} \right) &= 0. \end{aligned} \quad (72)$$

Note, that the obtained ODE systems are very similar, but simpler than the self-similar one. The first equation can be integrated, therefore a direct connection among $f, g,$ and h can be evaluated. (Our decade-long experience shows that, in most cases, the continuity equation can be integrated, which is a clear fingerprint of the mass conservation law.) The sum of the second and third equation can be integrated as well; after some additional algebra, a decoupled second-order non-linear ODE can be derived for the shape function of the density

$$2ff'' - \frac{f'^2}{4} + \frac{5}{2}c_0^2f^2 = 0. \quad (73)$$

The solution can be evaluated with quadrature and reads

$$f = \left(c_1 \sin \left[\frac{\sqrt{70}}{8} c_0 \eta \right] + c_2 \cos \left[\frac{\sqrt{70}}{8} c_0 \eta \right] \right)^{\frac{8}{7}}. \quad (74)$$

Fig. 8 presents the solution for a given parameter set. It is evident that the rational exponent of the sum of sine and cosine functions can only be defined for positive function values. So the fluid density function (which is $f(\eta)$) is defined on an infinite number of compact intervals. Note, that the first derivative of the density at the support remains finite which is good news, (infinite first derivatives would be problematic as a solution for a second order ODE).

Both self-similar and traveling-wave density solutions show the property of infinite number of zeros which is highly unusual, and which we have never examined in classical fluid mechanics. We think that this is a clear fingerprint of the deeper quantum mechanical property of these flow systems.

Fig. 9 shows the spatial and time dependence of the fluid density where the $y = 0$ coordinate projection was made. The result is a nice school example of a traveling-wave, the solution is periodic both in space and time.

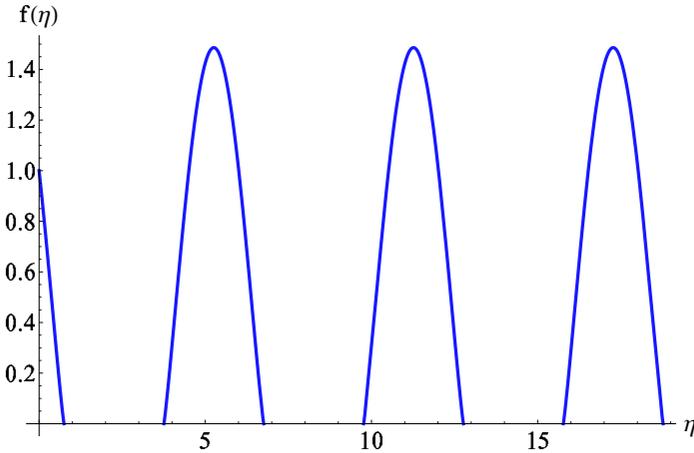


Figure 8. The solution of (74) for $c_1 = c_2 = c_0 = 1$.

We made hard efforts and tried to solve this Madelung system with the traveling-profile Ansatz as well. For more technical details see [52]. This is an interpolating formula between the self-similar and the traveling wave; for one variable and one spatial dimension, it has the form of $u(x, t) = a(t)f\left(\frac{x-b(t)}{c(t)}\right)$ where $a(t)$, $b(t)$, and $c(t)$ are continuous time-dependent functions with existing first-time derivatives. We applied the complete Ansatz of

$$\begin{aligned}\rho(x, y, t) &= a(t)f\left(\frac{x + y + b(t)}{c(t)}\right) = a(t)f(\eta), \\ u(x, y, t) &= d(t)g(\eta), \quad v(x, y, t) = e(t)h(\eta),\end{aligned}\tag{75}$$

unfortunately, the numerous constraints cannot be fulfilled to derive a time-independent ODE system for the shape functions $f(\eta)$, $g(\eta)$, and $h(\eta)$. The reader may think about more sophisticated Ansätze and try to solve the PDE system.

Finally, we have to mention that these results obtained from applying traveling-wave Ansatz are not published elsewhere.

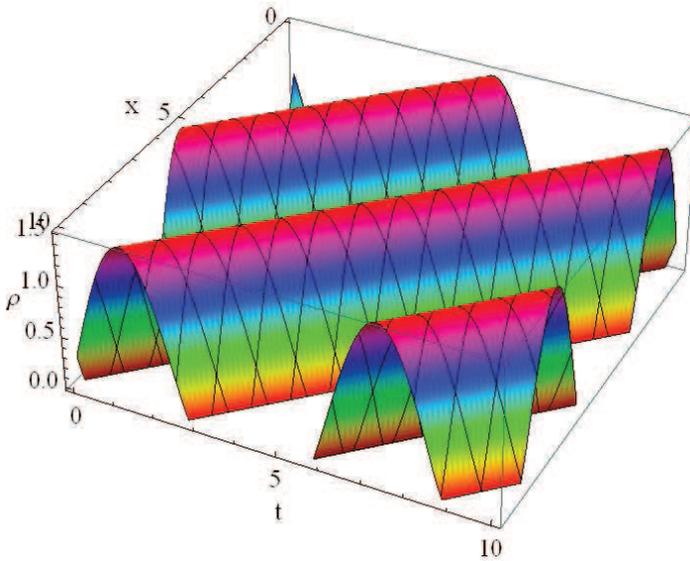


Figure 9. The time and space dependence of the fluid density function for the parameters mentioned above.

6. INVESTIGATION OF THE NON-LINEAR SCHRÖDINGER EQUATION

The Schrödinger equation of a free particle is one of the simplest quantum dynamical problems. There are numerous ways by which this problem can be complicated, including additional effects, interactions, or parameters. The simplest way is the investigation of the motion in a spherical potential. An additional step is including an external magnetic field with the minimal coupling and finally considering the role of the spin degree of freedom. As we mentioned in the introduction, the Madelung equations of such systems were derived in the '50s and '60 by Jánossy [22] which we will not discuss now. However, all such single-particle quantum systems remain linear with respect to the wave function.

We investigate a more interesting question, the free non-linear (or Gross-Pitaevskii) equation in the following.

In a weak-interacting Bose system, the macroscopic wave function Ψ appears as the order parameter of the Bose-Einstein condensation, obeying the

Gross-Pitaevskii (GP) equation [53, 54, 55, 56]

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m} + n|\Psi|^2 - \mu \right) \Psi. \quad (76)$$

Here, $n = 4\pi\hbar^2 a/m$ represents the coupling constant characterized by the s-wave scattering length a . The particle mass and the chemical potential are m and μ , respectively. Due to the second (non-linear) term of the right-hand side of this equation, it is definitely different from the linear Schrödinger equation (as was given above), therefore the following analysis with the presented results are independent as well.

The role and the properties of the GP system have been investigated from various viewpoints. Without completeness we mention some of them. The question of collective excitations of the trapped Bose gases was analyzed by [57]. Later, it is considered in the study of self-organization of the Bose-Einstein (BE) condensates [58].

Certain generalized versions of (76) have also been applied to the study of BE condensate of polaritons [59, 60, 61, 62, 63, 64, 65, 66, 67].

Let's consider the Madelung Ansatz for the order parameter with the amplitude and phase of $\Psi = \sqrt{\rho}e^{i\phi}$. The density part of the condensate is defined as ρ , and the superfluid-velocity part is defined via $\mathbf{v} = (\hbar/m)\nabla\phi$. (The expression "superfluid-velocity" is taken from Tsubota *et al.* [68] and means, in this context, that the fluid has no viscosity.)

The obtained hydrodynamical equations, the continuity and the Euler, are the following:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= \frac{\hbar^2}{2m^2} \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{\nabla}{m\rho} \left(\frac{n\rho^2}{2} \right), \end{aligned} \quad (77)$$

where we consider free motion (no extra potential energy term $U(\mathbf{r})$ is used) and the chemical potential μ was set to zero as well. It is easy to show that an additional chemical potential term μ gives no contribution to the final hydrodynamical equations. The role of the external potential energy term $U(\mathbf{r})$ will be mentioned later.

The first term on the right-hand side of the second equation is the quantum pressure term. The second term makes the difference to the original Madelung

equations; this is a kind of *second quantum pressure* due to the non-linear origin of the GP equation. Note, that these are most general vectorial equations for the velocity field \mathbf{v} which means that one-, two-, or three-dimensional motions can be investigated as well. In the following, we will consider the two-dimensional flow motion $\mathbf{v} = (u, v)$ in Cartesian coordinates. The functional form of the three- and one-dimensional equation system will be mentioned briefly, too.

6.1. Self-Similar Analysis

We search the solution of (77) in the form of our former self-similar Ansatz

$$\rho(x, y, t) = t^{-\alpha} f\left(\frac{x+y}{t^\beta}\right) := t^{-\alpha} f(\eta), \quad u(x, y, t) = t^{-\delta} g(\eta), \quad v(x, y, t) = t^{-\epsilon} h(\eta), \tag{78}$$

where $f, g,$ and h are the shape functions of the density and the velocity field, respectively. The similarity exponents $\alpha, \beta, \delta, \epsilon$ are of primary physical importance since α, δ, ϵ represent the damping of the magnitude of the shape function. Apart from some extraordinary cases, only all positive similarity exponents correspond to physically relevant dispersive solutions, which means decaying features at $x, y, t \rightarrow \infty$. Usually, negative exponents describe exploding solutions. Substituting the Ansatz (78) into (77) and executing additional algebraic manipulation, the following ODE system can be written for the shape functions

$$\begin{aligned} -\alpha f - \frac{\alpha}{2} f' \eta + f' g + f g' + f' h + f h' &= 0, \\ -\left(\frac{3}{2}\alpha - 1\right) g - \frac{\alpha}{2} g' \eta + g g' + h g' &= \frac{\hbar^2}{2m^2} \left(\frac{f'^3}{2f^3} - \frac{f' f''}{f^2} + \frac{f'''}{2f}\right) - \frac{n}{m} f', \\ -\left(\frac{3}{2}\alpha - 1\right) g - \frac{\alpha}{2} h' \eta + g h' + h h' &= \frac{\hbar^2}{2m^2} \left(\frac{f'^3}{2f^3} - \frac{f' f''}{f^2} + \frac{f'''}{2f}\right) - \frac{n}{m} f', \end{aligned} \tag{79}$$

where prime means derivation with respect to the new variable η . The relations among the similarity exponents are the following: α can be taken arbitrary, $\beta = \alpha/2$, and $\epsilon = \delta = 3\beta - 1 = 3\alpha/2 - 1$. Here we have to emphasize that a particular form of the external potential $U(x, y, t)$ can be added to (76) without leading to contradiction among the similarity exponents. The form in 2D is the following: $U(x, y, t) = \frac{1}{t^{2\beta}} \cdot v(\eta)$ where $v(\eta)$ can be any kind of function with existing first derivative.

(Of course, there is no evidence that the obtained modified (79) could be solved with analytical means.)

Note, that the ODE system of the linear SE (58) has a much simpler form where all the similarity exponents have the fixed value of $+1/2$. Our experience shows that this is usual for regular heat conduction, diffusion, or for non-compressible Navier-Stokes equations [43]. For the linear case, after some additional trivial algebraic steps, a decoupled ODE can be derived for the shape function of the density (59)

$$2f''f - (f')^2 + \frac{m^2\eta^2 f^2}{2\hbar^2} = 0. \quad (80)$$

The most general equation of (79) cannot be fully integrated by analytical means. The first ODE of such systems [43] explicitly means mass conservation in hydrodynamics. Furthermore, in most cases, it can be integrated analytically. However, the first equation of (79) is not a total derivative for any α . This is highly unusual among our investigated systems until now. As a second case, let's consider $-(3/2\alpha - 1) = \alpha/2$ this gives $\alpha = 1, \beta = \delta = \epsilon = 1/2$. Now the sum of the second and third equation of (79) can be integrated leading to

$$\frac{g^2}{2} + g\left(h - \frac{\eta}{2}\right) + \frac{h^2}{2} + \frac{h\eta}{2} + \frac{2nf}{m} - \frac{\hbar^2}{2m^2} \left(-\frac{f'^2}{4f^2} + \frac{f''}{2f}\right) + c_1 = 0. \quad (81)$$

For simplicity's sake, we set c_1 to zero. Note, that the variables g, f , and h are still coupled in this single equation. However, (81) is quadratic in $g(\eta)$ and the solutions can be expressed with the well-know formula of

$$g_{1,2} = \frac{\eta}{2} - h \pm \sqrt{\frac{\eta^2}{4} - \frac{4nf}{m} + \frac{\hbar^2}{2m^2} \left(-\frac{f'^2}{4f^2} + \frac{f''}{2f}\right)}. \quad (82)$$

Even this includes both functions of h and f . To avoid this problem, let's fix the discriminant to zero, which means that the velocity shape function will be single valued. The remaining ODE reads

$$2ff'' - (f')^2 + f^2 \left(\frac{\eta^2 m^2}{2\hbar^2} - \frac{8nmf}{\hbar}\right) = 0 \quad (83)$$

with the additional restraint that $4f^2 \neq 0$ meaning that the physical density of a fluid must be positive. Note, that without the last term we get (80) with the

solution of (60). We tried numerous variable transformations, unfortunately, we could not find any closed solution for the full (83). (Note, that if the coefficient of the second term would be twice as large as the first one, then an analytic solution would exist for the whole equation including the third order term as well.) It is also clear from (82) that the velocity shape functions have the trivial form of $g + h = \eta/2 = \frac{x+y}{2\sqrt{t}}$ which is the same as for the linear case.

Fig. 10 shows the analytical solution of (60) and numerically solution of (83) for various non-linear parameters n with the same initial conditions of $f(0) = 1$ and $f'(0) = 0$. The $\hbar = m = 1$ units are used. All solutions have a strong damping with stronger and stronger oscillations at large arguments. The functions are positive for all values of the argument (which is physical for a fluid density), but such oscillatory profiles are completely unknown in regular fluid mechanics [43]. The most interesting feature is the infinite number of zero values which cannot be interpreted physically for a classical real fluid. Remembering the restraint $4f^2 \neq 0$ of the calculation, the solution function falls into an infinite number of finite intervals. We think that this is a clear fingerprint that the obtained Euler equations have a quantum mechanical origin.

The major difference to the analytic solution is the following the $\int_0^\infty f(\rho)d\rho$ can be evaluated (now, of course, just numerically up to an arbitrary ρ_{max}) giving us a convergent integral. This clearly shows that the wave function of the corresponding quantum mechanical system (see below) is the element of the L^2 space. We investigated numerous initial conditions all giving the same result of a finite integral. Of course, this is just an empirical statement and not a rigorous mathematical proof.

The $\sqrt{\rho}$ function could be interpreted as the fluid mechanical analogue of the real part of the wave function of the free GP quantum mechanical particle. For the linear case, the square root of (60) is a kind of far analogue of the original free Gaussian wave packet. Unfortunately, we cannot find any direct transformation between these two functions.

To obtain the complete original wave function, the imaginary part has to be evaluated as well. It is simple from $\eta = \frac{x+y}{t^{1/2}} = 2(g + h)$ that

$$S = \frac{m}{\hbar} \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{v} d\mathbf{r} = \frac{m}{\hbar} \frac{(x + y)^2}{4t}. \tag{84}$$

Now

$$\Psi(x, y, t) = \sqrt{\rho(x, y, t)} e^{iS(x, y, t)} = \sqrt{t^{-1} \cdot f\left(\frac{[x + y]}{t^{1/2}}\right)} e^{\frac{im}{\hbar} \frac{(x+y)^2}{4t}} \tag{85}$$

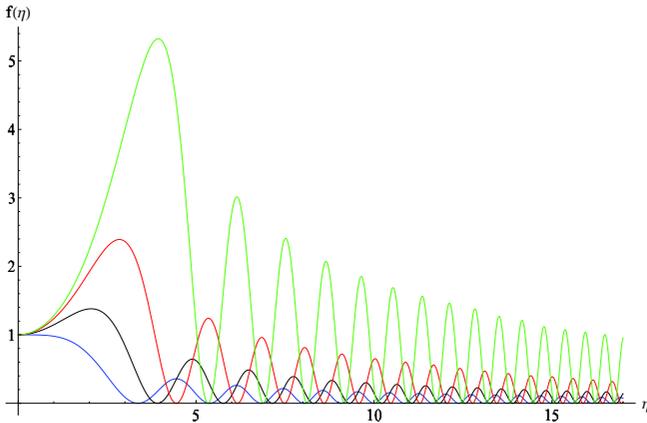


Figure 10. Various shape functions of the density. The blue curve is the analytic solution of (60) with $n = 0$ nonlinearity parameter, Black, red, and green function are numerical results of (83) for $n = 0.078$, 0.11 and 0.13 nonlinearity parameters. (All curves are for the same initial conditions of $f(0) = 1$ and $f'(0) = 0$.) Note, that larger parameters mean larger deviation from the analytic result.

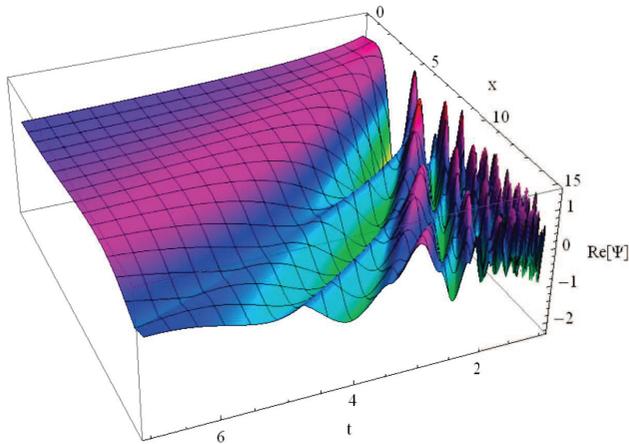


Figure 11. The projection of the real part of the wave function (85) along the $x - t$ plane.

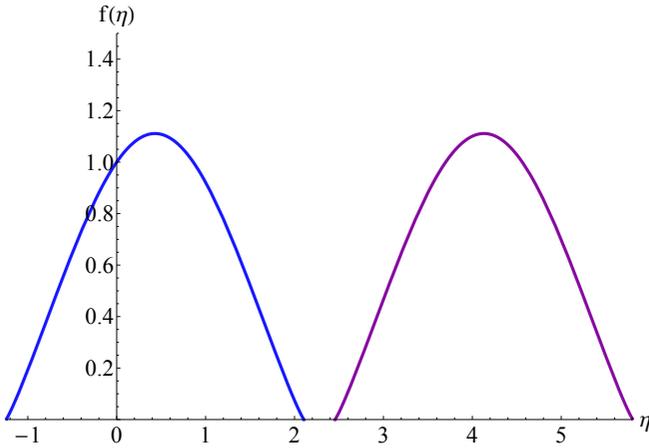


Figure 12. The shape function of the density, the numerical solution of (87). Note, the two disjunct parts of the solution on the presented interval. Two different initial condition sets were applied $f(0) = 1, f'(0) = 0.5$ for the left and $f(3.7) = 1, f'(3.7) = 0.5$ for the right part.

where $f(\eta)$ is now a numerical function. Fig. 11 shows the projection of the real part of the wave function to the x, t sub-space. At small times, there are clear oscillations, but at larger times the strong damping is evident. If the time is fixed ($t = t_0$ so to say) there are an infinite number of spatial oscillations which can be interpreted as a quantum mechanical heritage of the system. These results are published in our former paper of [70].

6.2. Traveling-Wave Analysis

To have a consistent and complete analysis of the non-linear SE as well, here we outline the obtained results from the traveling-wave Ansatz. We used the (71) form for the non-linear SE and obtained a slight modification of (72)

$$\begin{aligned}
 cf' + f'g + fg' + f'h + fh' &= 0, \\
 cg' + gg' + hg' - \frac{\hbar^2}{2m^2} \left(\frac{f'^3}{2f^3} - \frac{f'f''}{f^2} + \frac{f'''}{2f} \right) - \frac{n}{m}f' &= 0, \\
 ch' + gh' + hh' - \frac{\hbar^2}{2m^2} \left(\frac{f'^3}{2f^3} - \frac{f'f''}{f^2} + \frac{f'''}{2f} \right) - \frac{n}{m}f' &= 0. \quad (86)
 \end{aligned}$$

Applying the formerly mentioned algebraic manipulations, an independent decoupled ODE can be derived for the density shape function which contains an extra term compared to the linear one (73)

$$2ff'' - \frac{f'^2}{4} + \frac{5}{2}c_0^2f^2 + f^3 = 0. \quad (87)$$

We found no analytic solutions, therefore we investigated it with numerical means. Due to the last third modification term, the solutions remain very similar to the linear case. Fig. (12) presents some part of the density function. To evaluate disjunct parts of the solution, two different initial condition sets were applied.

SUMMARY AND OUTLOOK

In this chapter, we showed two possible viewpoints which may help the reader understand the meaning and the world of the SE.

First, we derived how the solution of the free SE can be obtained from a classical diffusion equation with the separation of variables which is the traditional method solving PDEs in transport phenomena. We solved the free SE with self-similar or traveling-wave Ansätze as well. (The physical picture behind these two trial functions are explained in detail.) All three solutions are interrelated, and it is instructive to see them in a common context. As a second, but unusual viewpoint, we introduced and analyzed the Madelung analogue (or form) of the SE. First, we gave a short historical review of the interpretation and development of the Madelung equation. In this picture, the SE is transformed to an ideal (non-viscous) fluid system where the dynamical variables are the fluid density and fluid velocity fields.

We received astonishing results: the fluid density distribution has a very peculiar property, namely highly oscillatory with an infinite number of zeros. To our knowledge, such behavior is not yet been noted in hydrodynamical systems of real fluids. We think that it is a clear fingerprint of the quantum mechanical origin of the system. We also believe that the noticeable quadratic argument of the solutions might be interpreted in the future. Lastly, we investigated the self-similar solution of the non-linear SE and compared it to the linear one. Our experiences are similar to the linear case. Work is in progress to analyze more complex or compound Madelung systems and obtain physically reasonable analytic results.

APPENDIX

Now, we teach the reader how the self-similar Ansatz leads to the final ODE system step-by-step. In two Cartesian space dimension, the Madlung PDE system reads as follows:

$$\begin{aligned} \rho_t + \rho_x u + \rho u_x + \rho_y v + \rho v_y &= 0, \\ u_t + uu_x + vu_y - \frac{\hbar^2}{2m} \nabla \left(\frac{\nabla^2}{\sqrt{\rho}} \right) &= 0, \\ v_t + uv_x + vv_y - \frac{\hbar^2}{2m} \nabla \left(\frac{\nabla^2}{\sqrt{\rho}} \right) &= 0, \end{aligned} \tag{88}$$

(for a better transparency we now use subscripts for partial derivatives) after performing the operations on the right-hand side

$$\begin{aligned} \rho_t + \rho_x u + \rho u_x + \rho_y v + \rho v_y &= 0, \\ u_t + uu_x + vu_y - \frac{\hbar^2}{2m} \left(\frac{\rho_x^3}{2\rho^3} - \frac{\rho_x \rho_{xx}}{\rho^2} + \frac{\rho_x xx}{2\rho} \right) &= 0, \\ v_t + uv_x + vv_y - \frac{\hbar^2}{2m} \left(\frac{\rho_y^3}{2\rho^3} - \frac{\rho_y \rho_{yy}}{\rho^2} + \frac{\rho_y yy}{2\rho} \right) &= 0. \end{aligned} \tag{89}$$

The applied Ansatz is

$$\rho(x, y, t) = t^{-\alpha} f\left(\frac{x+y}{t^\beta}\right) := t^{-\alpha} f(\eta), \quad u(x, y, t) = t^{-\delta} g(\eta), \quad v(x, y, t) = t^{-\epsilon} h(\eta). \tag{90}$$

All the time and spatial derivatives have to be evaluated

$$\begin{aligned} \rho_t &= -\alpha t^{-\alpha-1} f - \beta t^{-\alpha-1} f' \eta, & \rho_x &= t^{-\alpha-\beta} f' \\ u_t &= -\delta t^{-\delta-1} g - \beta t^{-\delta-1} g' \eta, & u_x &= t^{-\delta-\beta} g', & u_{xx} &= t^{-\delta-2\beta} g'', & u_{xxx} &= t^{-\delta-3\beta} g''', \\ v_t &= -\epsilon t^{-\epsilon-1} h - \beta t^{-\epsilon-1} h' \eta, & v_x &= t^{-\epsilon-\beta} h', & v_{xx} &= t^{-\epsilon-2\beta} h'', & v_{xxx} &= t^{-\epsilon-3\beta} h''', \end{aligned} \tag{91}$$

where prime means derivation in respect to η (e.g., $f' = \frac{df(\eta)}{d\eta}$). Note, the properties of the second term in the time derivative which, has an extra argument dependence, this feature loads to all the later mathematical finesse of the obtained ODEs. This property comes from the chain rule of derivation. (The

traveling-wave Ansatz has a much simpler structure $f(x + ct) = f(\omega)$, $f_x = f'$ and $f_t = cf'$ where $f' = \frac{df(\omega)}{d\omega}$.)

All the corresponding derivatives (91) have to be substituted within the original PDE system (89). At first glance, the resulting equations look tremendous

$$\begin{aligned}
 & -\alpha t^{-\alpha-1} f - t^{-\alpha-1} \beta f' \eta + t^{-\alpha-\beta-\delta} f' g + \\
 & \quad t^{-\alpha-\beta-\delta} f g' + t^{-\alpha-\beta-\epsilon} f' h + t^{-\alpha-\beta-\epsilon} f h' = 0, \\
 & -\delta t^{-\delta-1} g - t^{-\delta-1} \beta g' \eta + t^{-2\delta-\beta} g g' + t^{-\beta-\delta-\epsilon} h g' = \\
 & \quad \frac{\hbar^2}{2m^2} \left(\frac{(t^{-\alpha-\beta} f')^3}{2(t^{-\alpha} f)^3} - \frac{t^{-2\alpha-3\beta} f' f''}{(t^{-\alpha} f)^2} + \frac{t^{\alpha-3\beta} f'''}{2t^{-\alpha} f} \right), \\
 & -\epsilon t^{-\epsilon-1} h - t^{-\epsilon-1} \beta h' \eta + t^{-\delta-\beta-\epsilon} g h' + t^{-\beta-2\epsilon} h h' = \\
 & \quad \frac{\hbar^2}{2m^2} \left(\frac{(t^{-\alpha-\beta} f')^3}{2(t^{-\alpha} f)^3} - \frac{t^{-2\alpha-3\beta} f' f''}{(t^{-\alpha} f)^2} + \frac{t^{\alpha-3\beta} f'''}{2t^{-\alpha} f} \right).
 \end{aligned} \tag{92}$$

To have an ODE system which depends on the new variable η then all the time-dependent factors have to be eliminated, which means that all the exponents have to be the same for each ODE, therefore the corresponding algebraic equation system has to have a definite solution. Consequently, there are three possible cases:

- The system is overdetermined and contradictory to the exponents, so no possible self-similar solution exists for the corresponding PDE system. It also means that the physical system has no dissipative solutions of this type. Our decade-long experience demonstrates that such systems are very rare, but occur sometimes. For example, the telegraph equation has no self-similar solution; the very similar Euler-Poisson-Darboux equation, however, has such solutions. See [36].
- The system has one unique solution for every single exponent. (This is the present case, the linear SE.) It is nice, but the possible mathematical structure of the obtained ODE and the solutions are very restrictive.
- The system is under-determined, which usually means that one exponent remains free, which leads to the most complex mathematical structure of the solutions. For compressible fluids, the remaining free exponent may be chosen in such a way that an exponent in equation of states remains free, which means that various different materials with different physical properties can be analyzed. An equation of state in this context is a $p =$

$F(\rho)$ where the fluid pressure can be an arbitrary function of the fluid density. A good example for an under-determined system is [[38]].

In the present case (for the linear SE) we have

$$\begin{aligned}\alpha + 1 &= \alpha + \beta + \delta = \alpha + \beta + \epsilon, \\ \delta + 1 &= 2\delta + \beta = \epsilon + \delta + \beta = 3\beta, \\ \epsilon + 1 &= \epsilon + \delta + \beta = 2\epsilon + \beta = 3\beta.\end{aligned}\tag{93}$$

After some elementary algebraic manipulation, we get $\alpha = \beta = \delta = \epsilon = 1/2$. This analysis is very similar to other PDE systems as well [71].

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