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Travelling-Wave Solutions Of The Kardar-Parisi-Zhang Interface Growing Equation With Different Kind Of Noise Terms

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Abstract. The Kardar-Parisi-Zhang dynamic interface growth equation with the traveling-wave Ansatz is analyzed for one Cartesian space dimension. As a new feature the role of additional analytic terms are investigated. From the mathematical point of view these terms, can be considered as various noise distribution functions. Six different cases are investigated among others Gaussian, Lorentzian, white or even pink noise. Analytic solutions are evaluated and discussed for all cases. All results are expressible with various special functions like Airy, Bessel, Mathieu or Whittaker functions showing a very rich mathematical structure with some common general characteristics. This study is the continuation of our former work, where the same physical phenomena was investigated with the self-similar Ansatz.

INTRODUCTION

Crystal growth or the dynamics of solidification fronts are scientific topics which attract much interest from decades. Basic physics of growing crystallines can be found in large number of textbooks like [1]. One of the simplest nonlinear generalization of the ubiquitous diffusion equation is the so called Kardar-Parisi-Zhang(KPZ) model obtained from Langevin equation

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u + \frac{\lambda}{2}(\nabla u)^2 + \eta(\mathbf{x}, t), \tag{1}$$

where u stands for the profile of the local growth [2]. The first term on the right hand side describes relaxation of the interface by a surface tension preferring a smooth surface. The second term is the lowest-order nonlinear term that can appear in the surface growth equation justified with the Eden model. The origin of this term lies in non-equilibrium. The last term is a Langevin noise which mimics the stochastic nature of any growth process and has a Gaussian distribution usually. In the last two decades numerous studies came to light about the KPZ equation. Without completeness we mention a couple of them. The basic physical background of surface growth can be found in the textbook of Barabási and Stanley [3]. Later, Hwa and Frey [4, 5] investigated the KPZ model with the usage of the renormalization group-theory and the self-coupling method which is an sophisticated method using Green's functions. Additional dynamical scaling forms of $C(x,t) = x^{-2\varphi}C(bx, b^z t)$ were considered for the correlation function (where φ, b and z are real constants). The field theoretical approach was by Lässig to derive and investigate the KPZ equation [6]. Kirecherbauer and Krug published a review paper [7] where the KPZ equation was derived from hydrodynamical equations using a general current density relation.

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Numerous models exist, which may lead to similar equations as the KPZ model, i.e., the interface growth of bacterial colonies [8]. More general interface growing models were developed based on the so-called Kuramoto-Sivashinsky (KS) equation which is similar to the KPZ model with and extra $-\nabla^4 u$ term on the right hand side of (1) [9].

Beyond these continuous models based on partial differential equations (PDEs) there are numerous purely numerical methods available to study diverse surface growth phenomena. Without completeness, we mention the kinetic Monte Carlo [10], Lattice-Boltzmann simulations [11] and the etching model [12].

THEORY

In general non-linear PDEs has no general mathematical theory which could help us to understand general features or to derive physically relevant solutions. Basically, there are two different trial functions (or Ansatz) which have well-founded physical interpretation. The first one is the traveling wave solution, which mimics the wave property of the investigated phenomena described by the non-linear PDE with the form of

$$u(x,t) = f(x \pm ct) = f(\omega) \tag{2}$$

where c means the velocity of the corresponding wave. Gliding and Kersner used the traveling wave Ansatz to investigate study numerous reaction-diffusion equation systems [13]. To describe pattern formation phenomena [14] the traveling waves Ansatz is a useful tool as well. Saarloos investigated front propagation into unstable states [15] where traveling waves play a key role.

This simple trial function can be generalized in numerous ways, eg. to $e^{-\alpha t} f(x \pm ct) := e^{-\alpha t} f(\omega)$ which describes exponential decay or to $g(t) \cdot f(x \pm c \cdot t) := g(t) f(\omega)$ which can even be a power law function of time as well. (At this point we have to note, that the application of these Ansätze to the KPZ eq. leads to the triviality of $e^{-\alpha t} = g(t) \equiv 1$.) In 2006 He and Wu developed the so-called exp-function method [16] which relies on an Ansatz (a rational combination of exponential functions), The second physically relevant Ansatz is the self-similar one which desribes the dispersive characteristics of the investigated phenomena. In our present study we use 2.

THE SUMMATION OF THE RESULTS

First we investigated the travelling-wave solution of the KPZ equation without any additional noise term and compared it to the self-similar solution.

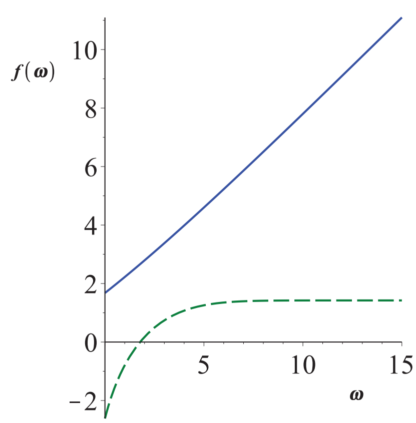

FIGURE 1. The different shape functions of the KPZ equation without noise term. The solid line represents the solution for traveling-wave and the dashed line is for the self-similar Ansatz. The applied parameter set is $c_1 = c_2 = c = 1,\ \nu = 4,\ \lambda = 3$

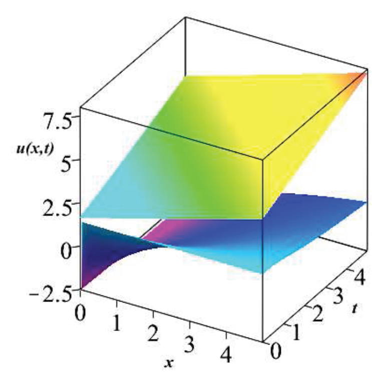

FIGURE 2. The different solutions of the KPZ equation without any kind of noise term. The upper lying function represents the traveling-wave solution. The applied parameter set is the same as used above.

Remark that the solution to (1) obtained from the self-similar Ansatz reads

$$f(\omega) = \frac{2\nu}{\lambda} ln\left(\frac{\lambda c_1 \sqrt{\pi\nu}\ erf[\omega/(2\sqrt{\nu})] + c_2}{2\nu}\right), \tag{3}$$

where $erf[\ ]$ means the error function [18].

Figure 1 and 2 show the complete solutions in 1D ($f(\omega)$) as the function of ω and in 2D ($u(x,t)$) as the function of time and spatial coordinate. The lower surface represents the self-similar solution analysed in [17] and the higher lying plane is the travelling-wave solution. It is clear that without any kind of additional noise term $\eta(\omega[x,t])$ the surface growing is infinite and no extra structure is present.

In our in-depth analysis the effects of additional six different kind of noise terms were investigated. Four of them are power-law kind of noises and the additional two are the Lorenzian and the periodic noise, respectively. The next enumeration presents the direct form of the noise terms and the obtained functions in the results:

- The $\eta = a/\omega^2$ brown noise resulting an expression with modified Bessel functions $I(\omega)$ and $K(\omega)$.
- The $\eta = a/\omega$ pink noise resulting an expression with the Kummer functions $M(\omega)$ and $U(\omega)$
- The $\eta = constant$ white noise results a sum of a linear function of ω and the $\ln(f(1/e^{\omega}))$
- The $\eta = a\omega$ blue noise results an expression with the Airy functions of $Ai(\omega)$ and $Bi(\omega)$
- The $\eta = \frac{a}{1+\omega^2}$ Lorenzian noise results an expression which the Heun functions $H(\omega)$
- The $\eta = a\sin(\omega)$ preiodic noise results an expression with the Mathieu functions $S(\omega)$ and $C(\omega)$.

Let us see an example of noise term cases. We consider the brown noise $\eta(x,t) = \frac{a}{\omega^2}$. It leads to the following ODE

$$-\nu f''(\omega) + f'(\omega)\left[c - \frac{\lambda}{2} f'(\omega)\right] - \frac{a}{\omega^2} = 0. \tag{4}$$

The solution can be given in the form

$$f(\omega) = \frac{1}{\lambda}\left(c\eta + \nu \ln\left\{\frac{\lambda^2\left[-c_1 I_d\left(\frac{c\omega}{2\nu}\right) + c_2 K_d\left(\frac{c\omega}{2\nu}\right)\right]^2}{c^2\omega\left[K_d\left(\frac{c\omega}{2\nu}\right) I_{d+1}\left(\frac{c\omega}{2\nu}\right) + I_d\left(\frac{c\omega}{2\nu}\right) K_{d+1}\left(\frac{c\omega}{2\nu}\right)\right]^2}\right\}\right) \tag{5}$$

where $I_d(\omega)$ and $K_d(\omega)$ are the modified Bessel functions of the first and second kind [18] with the subscript of $d = \frac{\sqrt{\nu^2 - 2a\lambda}}{2\nu} + 1$.

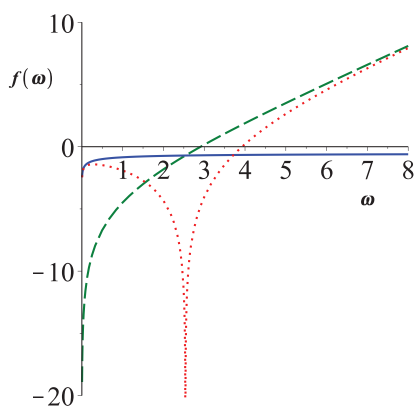

FIGURE 3. Three different shape functions. The physical parameter set is $\lambda = 5, \nu = 3, a = 2$ and $c = 2$. The dashed line is for $c_1 = 1, c_2 = 0$, the dotted line is for $c_1 = c_2 = 1$ and the solid line is for $c_1 = 0, c_2 = 1$, respectively.

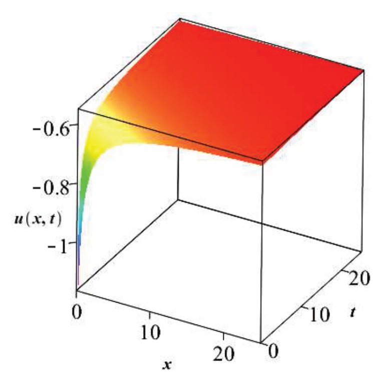

FIGURE 4. The solution $u(x,t)$ to the KPZ equation for the brown noise with the parameter set of $c_1 = c_2 = c = 1$, $\nu = 4$, $\lambda = 3$.

To obtain real solutions for the KPZ equation (which provides the height of the surface) the order of the Bessel function (notated as the subscript) has to be non-negative and provides the following constrain $\nu^2 \geq 2a\lambda$. This gives us a reasonable relation among the three terms of the right hand side of equation (1). When the magnitude of the noise term a becomes large enough no surface growth take place.

Figure 3 presents solutions with different combinations of the integration constants c_1, c_2. Having in mind, that the $K_d()$ Bessel function of the second kind is regular at infinity, one gets that it has a strong decay at large argument ω. The $c_1 = 0, c_2 = 0$ type solutions have physical relevance. Figure 4 shows the complete solution of the KPZ equation. It can be seen that a sharp and localized peak exists for a short time. Therefore, no typical surface growth phenomena is described with this kind of noise and initial conditions.

All the physical properties of the results are analysed and the role of the free physical parameters like ν, λ, c, a are discussed.

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REFERENCES

[1] Y. Saito, *Statistical Physics of Crystal Growth,* World Scientific Press 1996
[2] M. Kardar, G. Parisi and Yi-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[3] A.-L. Barabási, *Fractal concepts in surface growth*, Press Syndicate of the University of Cambridge, New York, NY, USA, 1995.
[4] T. Hwa and E. Frey, Phys. Rev. A **44**, R7873 (1991).
[5] E. Frey, U.C. Täubner and T. Hwa, Phys. Rev. E. **53**, 4424 (1996).
[6] M. Lässig, J. Phys: Condens. Matter. **10**, 9905 (1998).
[7] T. Kriecherbauer and J. Krug, J. Phys. A: Math. Theor. **43**, 403001 (2010).
[8] M. Matsushita, J. Wakita, H. Itoh, I. Rafols, T. Matsuyama, H. Sakaguchi, M. Mimura, Physica A **249**, 517 (1998).
[9] Y. Kuramoto and T. Tsuzki, Prog. Theor. Phys. **55**, 356 (1976), G.I. Sivashinsky, Physica D, **4**, 227 (1982).
[10] T. Martynec and S. H. L. Klapp, Phys. Rev. E **98**, 042801 (2018).
[11] D. Sergi, A. Camarano, J. M. Molina, A. Ortona and J. Narciso, International Journal of Modern Physics C, **27**, 1650062 (2016).
[12] B. A. Mello, Physica A: Statistical Mechanics and its Applications, **419**, 762 (2015).
[13] B.H. Gilding and R. Kersner, *Travelling Waves in Nonlinear Diffusion-Convection Rea ctions*, Progress in Nonlinear Differential Equations and Their Applications, Birkhauser Verlag, Basel-Boston-Berlin, 2004.
[14] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **85**, 851 (1993).
[15] W. von Saarloos, Phys. Rep. **386**, 29 (2003).
[16] J.H. He and X.H. Wu, Chaos, Solitons and Fractals **30**, 700 (2006).
[17] I. F. Barna, G. Bognár, M. Guedda, L. Mátyás and K. Hriczó, Math. Model. Anal., **25**, 2 (2020).
[18] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark *NIST Handbook of Mathematical Functions* Cambridge University Press, 2010.