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Sandra Pinelas · John R. Graef ·
Stefan Hilger · Peter Kloeden ·
Christos Schinas *Editors*

Differential and Difference Equations with Applications

ICDDEA 2019, Lisbon, Portugal, July 1–5

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Analytic Traveling-Wave Solutions of the Kardar-Parisi-Zhang Interface Growing Equation with Different Kind of Noise Terms



I. F. Barna, G. Bognár, L. Mátyás, M. Guedda, and K. Hriczó

Abstract The one-dimensional Kardar-Parisi-Zhang dynamic interface growth equation with the traveling-wave Ansatz is analyzed. As a new feature additional analytic terms are added. From the mathematical point of view, these can be considered as various noise distribution functions. Six different cases were investigated among others Gaussian, Lorentzian, white or even pink noise. Analytic solutions are evaluated and analyzed for all cases. All results are expressible with various special functions Mathieu, Bessel, Airy or Whittaker functions showing a very rich mathematical structure with some common general characteristics. This study is the continuation of our former work, where the same physical phenomena was investigated with the self-similar Ansatz. The differences and similarities among the various solutions are enlightened.

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1 Introduction

Solidification fronts or crystal growth is a scientific topic which attracts much interest from a long time. Basic physics of growing crystallines can be found in large number of textbooks (see e.g., [1]). One of the simplest nonlinear generalization of the ubiquitous diffusion equation is the so called Kardar-Parisi-Zhang (KPZ) model obtained from Langevin equation

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u + \frac{\lambda}{2} (\nabla u)^2 + \eta(\mathbf{x}, t), \quad (1)$$

where u stands for the profile of the local growth [2]. The first term on the right hand side describes relaxation of the interface by a surface tension preferring a smooth surface. The next term is the lowest-order nonlinear term that can appear in the surface growth equation justified with the Eden model. The origin of this term lies in non-equilibrium. The third term is a Langevin noise which mimics the stochastic nature of any growth process and usually has a Gaussian distribution. In the last two decades numerous studies came to light about the KPZ equation. Without completeness we mention some of them. The basic physical background of surface growth can be found in the book of Barabási and Stanley [3]. Later, Hwa and Frey [4, 5] investigated the KPZ model with the help of the renormalization group-theory and the self-coupling method which is a precise and sophisticated method using Green's functions. Various dynamical scaling forms of $C(x, t) = x^{-2\varphi} C(bx, b^z t)$ were considered for the correlation function (where φ , b and z are real constants). The field theoretical approach by Lässig was to derive and investigate the KPZ equation [6]. Kriecherbauer and Krug wrote a review paper [7], where the KPZ equation was derived from hydrodynamical equations using a general current density relation.

Several models exist and all lead to similar equations as the KPZ model, one of them is the interface growth of bacterial colonies [8]. Additional general interface growing models were developed based on the so-called Kuramoto-Sivashinsky (KS) equation which shows similarity to the KPZ model with an extra $\nabla^4 u$ term [9].

Kersner and Vicsek investigated the traveling wave dynamics of the singular interface equation [10] which is closely related to the KPZ equation. One may find certain kind of analytic solutions to the problem [11] as already mentioned in [12].

Ódor and co-worker intensively examined the two dimensional KPZ equation with dynamical simulations to investigate the aging properties of polymers or glasses [13].

Beyond these continuous models based on partial differential equations (PDEs), there are large number of purely numerical methods available to study diverse surface growth effects. As a view we mention the kinetic Monte Carlo [14] model, Lattice-Boltzmann simulations [15], and the etching model [16].

In this paper we investigate the solutions to the KPZ equation with the traveling wave Ansatz in one-dimension applying various forms of the noise term. The effects of the parameters involved in the problem are examined.

2 Theory

In general, non-linear PDEs has no general mathematical theory which could help us to understand general features or to derive physically relevant solutions. Basically, there are two different trial functions (or Ansatz) which have well-founded physical interpretation. The first one is the traveling wave solution, which mimics the wave property of the investigated phenomena described by the non-linear PDE of the form

$$u(x, t) = f(x \pm ct) = f(\omega), \quad (2)$$

where c means the velocity of the corresponding wave. Gliding and Kersner used the traveling wave Ansatz to investigate study numerous reaction-diffusion equation systems [17]. To describe pattern formation phenomena [18] the traveling waves Ansatz is a useful tool as well. Saarloos investigated the front propagation into unstable states [19], where traveling waves play a key role.

This simple trial function can be generalized in numerous ways, e.g., to $e^{-\alpha t} f(x \pm ct) := e^{-\alpha t} f(\omega)$ which describes exponential decay or to $g(t) \cdot f(x \pm c \cdot t) := g(t)f(\omega)$ which can even be a power law function of the time as well. We note, that the application of these Ansatz to the KPZ equation leads to the triviality of $e^{-\alpha t} = g(t) \equiv 1$. In 2006, He and Wu developed the so-called exp-function method [20] which relying on an Ansatz (a rational combination of exponential functions), involving many unknown parameters to be specified at the stage of solving the problem. The method soon drew the attention of many researchers, who described it as “straightforward”, “reliable”, and “effective”. Later, Aslan and Marinakis [21] summarized various applications of the Ansatz.

There is another existing remarkable Ansatz interpolating the traveling-wave and the self-similar Ansatz by Benhamidouche [22].

The second one is the self-similar Ansatz [23] of the form $u(x, t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right) := t^{-\alpha} f(\omega)$. The associated mathematical and physical properties were exhaustively discussed in our former publications [24, 25] or in a book chapter [26] in the field of hydrodynamics. All these kind of methods belong to the so-called reduction mechanism, where applying a suitable variable transformation the original PDEs or systems of PDEs are reduced to an ordinary differential equation (ODE) or systems of ODEs.

3 Results Without the Noise Term

Applying the traveling wave Ansatz to the KPZ PDE with $\eta(x, t) = 0$, Eq. (1) leads to the ODE of

$$-vf''(\omega) + f'(\omega) \left[c - \frac{\lambda}{2} f'(\omega) \right] = 0, \quad (3)$$

From now on we use the Maple 12 mathematical program package to obtain analytic solutions in closed forms. For Eq. (3), it can be given as

$$f(\omega) = \frac{2}{\lambda} \ln \left(\frac{\lambda \left[c_1 v e^{\frac{c\omega}{v}} + c_2 c \right]}{2vc} \right) v, \tag{4}$$

where c_1 and c_2 are the constants of integration and c is the speed of the wave.

We fix this notation from now on throughout the paper. Note, that this is an equation of a linear function $f(\omega) = a\omega + b$ (just given in a complicated form) with any kind of parameter set, except $c_1 = 0$ which gives a constant solution. This physically means that there is a continuous surface growing till infinity which is quite unphysical. Therefore, some additional noise is needed to have surface growing phenomena. We remark the general properties of all the forthcoming solutions. Due to the Hopf-Cole transformation [27, 28] ($h = A \ln(y)$) converts the non-linear KPZ equation to the regular heat conduction (or diffusion) equation with an additional stochastic source term eliminating the non-linear gradient-squared term. All the solutions contain a logarithmic function with a complicated argument. In this sense, the solutions have the same structure, the only basic difference is the kind of special function in the argument. If these argument functions take periodically positive and negative real values then the logarithmic function creates distinct intervals (small islands which describe the surface growing mechanisms, and define the final solution). This statement is generally true for our former study as well [29, 30].

Remark that the solution to (1) obtained from the self-similar Ansatz reads

$$f(\omega) = \frac{2v}{\lambda} \ln \left(\frac{\lambda c_1 \sqrt{\pi v} \operatorname{erf}[\omega/(2\sqrt{v})] + c_2}{2v} \right), \tag{5}$$

where $\operatorname{erf}[\]$ means the error function [31]. Figure 1 compares these two solutions. We note the asymptotic convergence of the self-similar solution and the divergence of the traveling-wave solution. We have the same conclusion as in our former study [29] (where the self-similar Ansatz was applied), that without any noise term the KPZ equation cannot be applied to describe surface growth phenomena. The different kind of noise terms define different kind of extra islands (parts of the solution having compact supports) and these islands show a growth dynamics.

To have a better understanding between the two solutions, Fig. 2 shows the projection of both complete solutions $u(x, y = 0, t)$. The major differences are still present.

Fig. 1 The two shape functions of the KPZ equation without any kind of noise term. The solid line represents the solution for traveling-wave and the dashed line is for the self-similar Ansatz. The applied parameter set is $c_1 = c_2 = c = 1, \nu = 4, \lambda = 3$

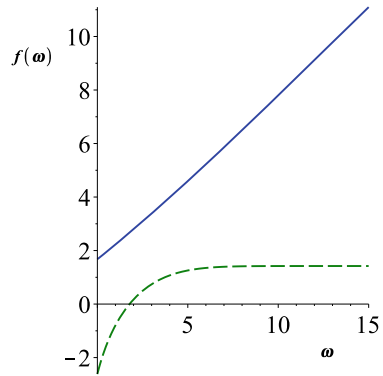
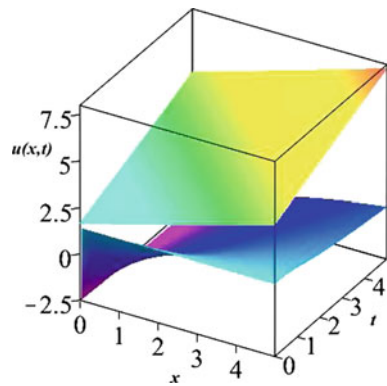


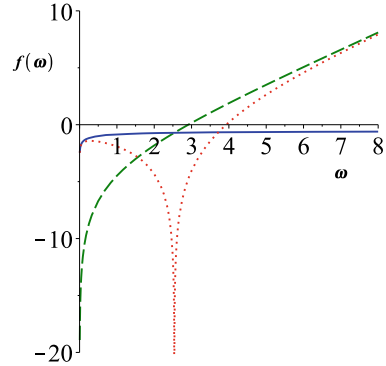
Fig. 2 The two solutions of the KPZ equation without any noise term. The upper lying function represents the traveling-wave solution. The applied parameter set is the same as used above



4 Results with Various Noise Terms

As we mentioned in our former study [29] only the additional noise term makes the KPZ solutions interesting. We search the solutions with the traveling-wave Ansatz, therefore is it necessary that the noise term η should be an analytic function of $\omega = x + ct$ like $\eta(\omega) = a(x + ct)^2$. We will see that for some kind of noise terms it is not possible to find a closed analytic solution when all the physical parameters are free (ν, λ, c, a), however, if some parameters are fixed it becomes possible to find analytic expressions. It is also clear, that it is impossible to perform a mathematically rigorous complete function analysis according to all four physical and two integral parameters c_1, c_2 . We performed numerous parameter studies and gave the most relevant parameter dependencies of the solutions.

Fig. 3 Three different shape functions for the brown noise $n = -2$. The applied physical parameter set is $\lambda = 5, \nu = 3, a = 2$ and $c = 2$. The dashed line is for $c_1 = 1, c_2 = 0$, the dotted line is for $c_1 = c_2 = 1$ and the solid line is for $c_1 = 0, c_2 = 1$, respectively



4.1 Brown Noise $n = -2$

As first, case let us consider the brown noise $\eta(x, t) = \frac{a}{\omega^2}$. It leads to the following ODE

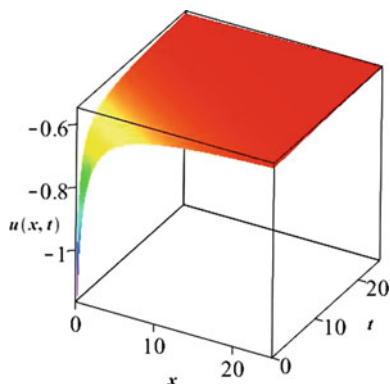
$$- \nu f''(\omega) + f'(\omega) \left[c - \frac{\lambda}{2} f'(\omega) \right] - \frac{a}{\omega^2} = 0. \tag{6}$$

The solution can be given in the form

$$f(\omega) = \frac{1}{\lambda} \left(c\eta + \nu \ln \left\{ \frac{\lambda^2 \left[-c_1 I_d \left(\frac{c\omega}{2\nu} \right) + c_2 K_d \left(\frac{c\omega}{2\nu} \right) \right]^2}{c^2 \omega \left[K_d \left(\frac{c\omega}{2\nu} \right) I_{d+1} \left(\frac{c\omega}{2\nu} \right) + I_d \left(\frac{c\omega}{2\nu} \right) K_{d+1} \left(\frac{c\omega}{2\nu} \right) \right]^2} \right\} \right) \tag{7}$$

where $I_d(\omega)$ and $K_d(\omega)$ are the modified Bessel functions of the first and second kind [31] with the subscript of $d = \frac{\sqrt{\nu^2 - 2a\lambda}}{2\nu} + 1$. To obtain real solutions for the KPZ equation (which provides the height of the surface) the order of the Bessel function (notated as the subscript) has to be non-negative and provides the following constrain $\nu^2 \geq 2a\lambda$. This gives us a reasonable relation among the three terms of the right hand side of Eq. (1). When the magnitude of the noise term a becomes large enough no surface growth take place. Figure 3 presents solutions with different combinations of the integration constants c_1, c_2 . Having in mind, that the $K_d()$ Bessel function of the second kind is regular at infinity, one gets that it has a strong decay at large argument ω . The $c_1 = 0, c_2 = 0$ type solutions have physical relevance. Figure 4 shows the complete solution of the KPZ equation. It can be seen that a sharp and localized peak exists for a short time. Therefore, no typical surface growth phenomena is described with this kind of noise and initial conditions.

Fig. 4 The solution $u(x, t)$ to the KPZ equation for the brown noise $n = -2$ with the parameter set of $c_1 = c_2 = c = 1, \nu = 4, \lambda = 3$



4.2 Pink Noise $n = -1$

The noise term $\eta = \frac{a}{\omega}$ corresponds to the ODE

$$- \nu f''(\omega) + f'(\omega) \left[c - \frac{\lambda}{2} f'(\omega) \right] - \frac{a}{\omega} = 0, \tag{8}$$

whose general solution is

$$f(\omega) = \frac{1}{\lambda} + \ln \left\{ \frac{-c\lambda[c_1 M(\varepsilon_b) - c_2 U(\varepsilon_b)]}{M(\varepsilon_d)(2\nu c U(\varepsilon_b) + a\lambda U(\varepsilon_b)) + 2M(\varepsilon_b)\nu c U(\varepsilon_d)} \right\}, \tag{9}$$

where $M(\varepsilon_b)$ and $U(\varepsilon_d)$ are the Kummer M and Kummer U functions (for more see [31]) with the parameters of $\varepsilon_b = (\frac{2c\nu - a\lambda}{2c\nu}, 2, \frac{c\omega}{\nu})$ and $\varepsilon_d = (\frac{-a\lambda}{2c\nu}, 2, \frac{c\omega}{\nu})$. Figure 5 shows three different shape functions corresponding to the pink noise. The evaluation of direct parameter dependencies of the solutions are not trivial. In some reasonable parameter range we found the following trends: for fixed a, c, ν and larger λ values, the solution shows more independent well-defined “bumps” or islands and higher steepness of the line which connects the maxima of the existing peaks of the islands. At fixed parameter values a, c, λ , different values of ν just shift the position of the existing peaks. The role of a and c is not defined. Figure 6 presents a total solution $u(x, t)$ to the KPZ equation, the freely traveling three islands are clearly seen.

4.3 White Noise $n = 0$

Here, the noise term is $\eta = a\omega^0 = a$ which leads to the ODE of

Fig. 5 Three different shape functions for the pink noise ($n = -1$). Solid, dashed and dotted lines are for the parameter sets of $(c_1 = c_2 = 1; c = 1/2, \nu = 0.85, \lambda = 3, a = 2)$, $(c_1 = c_2 = 1; c = 1/2, \nu = 0.85, \lambda = 2.5, a = 2)$, $(c_1 = c_2 = 1; c = 0.6, \nu = 0.85, \lambda = 5, a = 2)$, respectively

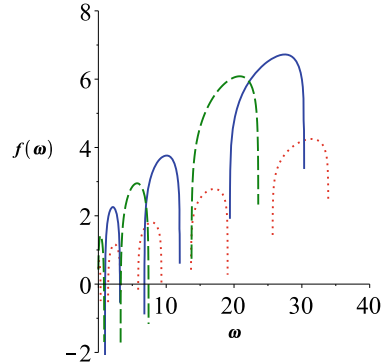
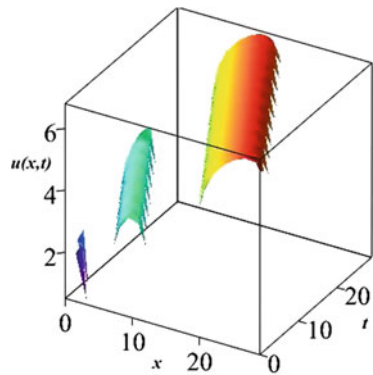


Fig. 6 The total solution of the KPZ equation for $n = -1$ with the applied parameter set $c_1 = c_2 = 1, c = 1/2, \nu = 0.85\lambda = 3$ and $a = 2$



$$- \nu f''(\omega) + f'(\omega) \left[c - \frac{\lambda}{2} f'(\omega) \right] - a = 0, \tag{10}$$

$$f(\omega) = \frac{\omega c}{\lambda} - \frac{\omega \sqrt{c^2 - 2a\lambda}}{\lambda} - \frac{2\nu \ln(2)}{\lambda} - \frac{\nu \ln \left(\frac{c^2 - 2a\lambda}{\lambda^2 \left[c_1 e^{\frac{\omega \sqrt{c^2 - 2a\lambda}}{\nu}} - c_2 \right]^2} \right)}{\lambda} \tag{11}$$

Figure 7 shows two shape functions for two different parameter sets. There exists basically two different functions depending on the ratios of the integral constants c_1 and c_2 . The first is a pure linear function with infinite range and its domain represents boundless surface growth, which is a physical nonsense. The second solution is a sum of a linear and logarithmic function with a domain bounded from above due to the argument of the \ln function. Figure 8 shows the final solution of the KPZ equation $u(x, t)$. We note that with the substitution $\omega = x + ct$ only the first kind of solution

Fig. 7 Two shape functions for the constant or white noise. The solid line is for the parameter set $c_1 = 4, c_2 = -1, c = 0.3, v = 2, \lambda = 1, a = 1$, and the dashed line is for $c_1 = c_2 = 1, c = 4, v = 0.5, \lambda = 1, a = 0.3$, respectively

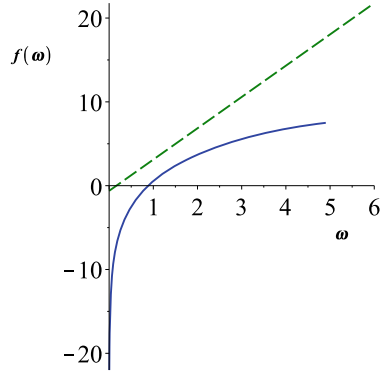
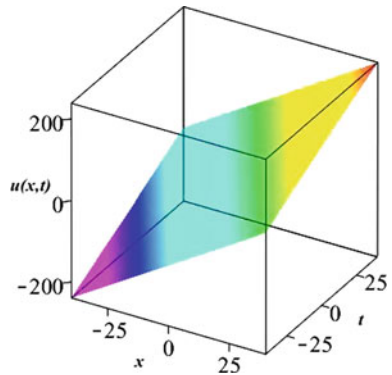


Fig. 8 The KPZ solution for the constant or white noise. The applied parameter set is $c_1 = c_2 = 1, c = 4, v = 0.5, \lambda = 1, a = 0.3$, respectively



remains real. For the second parameter set which creates a modified \ln function with a cut at well-defined argument becomes complex.

4.4 Blue Noise $n = 1$

The last color noise $\eta = a\omega$ leads to the ODE of

$$-vf''(\omega) + f'(\omega) \left[c - \frac{\lambda}{2} f'(\omega) \right] - a\omega = 0, \tag{12}$$

with the general solution of

$$f(\omega) = \frac{c\omega}{\lambda} - \frac{4v \ln(2)}{3\lambda} + \frac{2v}{3\lambda} \ln \left\{ \frac{\lambda^2 [c_1 Ai(\tilde{\omega}) - c_2 Bi(\tilde{\omega})]^3}{va [Ai(1, \tilde{\omega}) Bi(1, \tilde{\omega}) - Bi(1, \tilde{\omega}) Ai(\tilde{\omega})]^3} \right\} \tag{13}$$

Fig. 9 The shape function for the blue noise for three parameter sets. The solid, dashed and dotted lines are for the parameter sets $(c_1 = 1, c_2 = 0, c = 3, a = 0.5, \nu = 1.5, \lambda = 2)$, $(c_1 = c_2 = c = 1, a = 1, \nu = 1, \lambda = 3)$, and $(c_1 = c_2 = c = 1, a = 1, \nu = 1, \lambda = 3)$, respectively

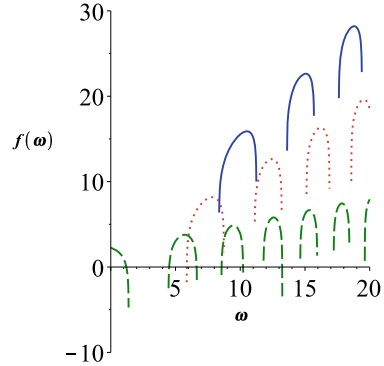
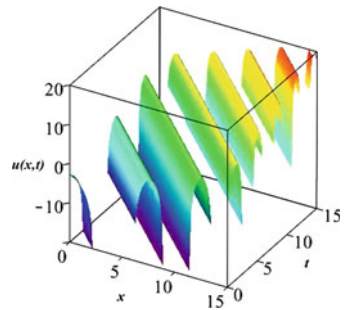


Fig. 10 The solution $u(x, t)$ for the $n = 1$ or blue noise with the applied parameter set of $c_1 = c_2 = c = 1, \nu = 2, \lambda = 3, a = 1$



where $Ai(\tilde{\omega}), Bi(\tilde{\omega})$ denote the Airy functions of the first and second kind and $Ai(1, \tilde{\omega})$ and $Bi(1, \tilde{\omega})$ are the first derivatives of the Airy functions, where we used the following notation: $\tilde{\omega} = \frac{-(2a\omega\lambda - c^2)4^{\frac{1}{3}}\left(\frac{a\lambda}{\nu^2}\right)^{1/3}}{4a\lambda}$. Exhaustive details of the Airy function can be found in [32]. When the argument ω is positive, $Ai(\omega)$ is positive, convex, and decreasing exponentially to zero, while $Bi(\omega)$ is positive, convex, and increasing exponentially. When ω is negative, $Ai(\omega)$ and $Bi(\omega)$ oscillate around zero with ever-increasing frequency and ever-decreasing amplitude.

Figure 9 represents shape functions with different parameter sets. Our analysis showed that the composite argument of the ln function is purely real having a decaying oscillatory behavior with alternatively positive and negative values. The ln function creates infinite number of separate “bumps” or islands with compact supports and infinite first spatial derivatives at their boarders. Combining the first two terms of the (13), we get an infinite series of separate islands with increasing height. The ratio c/λ is the steepness of the line, this automatically defines the steepness of the absolute height of the islands. The effects of the various parameters are not quite independent and hard to define, we may say that in general each parameter ν, λ, a, c alone can change the widths, spacing and absolute height of the peaks. Figure 10 shows the total solution of the KPZ equation. The traveling “bumps” are clearly visible.

4.5 Lorentzian Noise

As a first non-colour noise let us consider the Lorentzian noise of the form $\eta = \frac{a}{1+\omega^2}$. It leads to the ODE of

$$- \nu f''(\omega) + f'(\omega) \left[c - \frac{\lambda}{2} f'(\omega) \right] - \frac{a}{1 + \omega^2} = 0, \tag{14}$$

We mention, that for the classical exponential and Gaussian noise distributions we could not give solutions in closed analytic form. Unfortunately, there is no closed analytic expression available if all the parameters (ν, λ, c, a) are free. The formal solution contains integrals of the Heun C confluent functions multiplied by some polynomials. However, if the parameters a, λ, ν are fixed, there is analytic solution available for free propagation speed c . The exact solution for $a = \lambda = \nu = 1/2$, and $c = 2$ is the following

$$f(\omega) = c\omega + 2 \ln \left\{ \frac{c_1 C(B) - c_2 \omega C(A)}{2(\omega^4 + \omega^2)[C(A)C'(B) - C(B)C'(A)] + (1 + \omega^2)C(A)C(B)} \right\}, \tag{15}$$

where $C'()$ means the first derivative of the Heun C function [33]. For the better transparency we introduce the following notations $A = 0, \frac{1}{2}, 1, \frac{c^2}{4}, 1 - \frac{c^2}{4}; -\omega^2$ and $B = 0, -\frac{1}{2}, 1, \frac{c^2}{4}, 1 - \frac{c^2}{4}; -\omega^2$.

Figure 11 shows the shape function for given parameter set. There is a broad island close to the origin and numerous tiny ones at larger arguments. The numerical accuracy of Maple 12 was enhanced to reach this resolution. It is well-known that the Heun functions are the most complicated objects among special functions and the evaluations needs more computer time.

Figure 12 presents the total solution of the original KPZ. Due to the substitution $\omega = x + ct$ the original local solution broke down to several smaller islands which freely propagate in time and space.

4.6 Periodic Noise

The last perturbation investigated is a periodic function $\eta = a \sin(\omega)$ and

$$- \nu f''(\omega) + f'(\omega) \left[c - \frac{\lambda}{2} f'(\omega) \right] - a \sin(\omega) = 0. \tag{16}$$

Fig. 11 The shape function for the Lorentzian noise. The applied parameters are $c_1 = 0.5, c_2 = 2, c = 1, v = 1, a = 1, \lambda = 3$

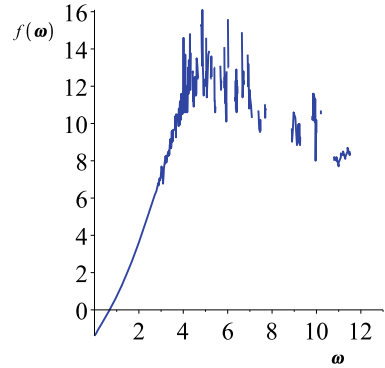
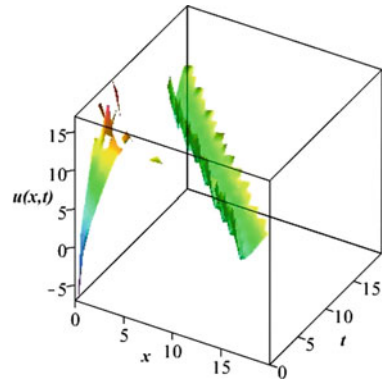


Fig. 12 The solution of the KPZ equation for Lorentzian noise, with the parameters mentioned above



The general solution can be given as

$$f(\omega) = \frac{1}{\lambda} \left(c\omega + 2 \ln \left\{ \frac{\lambda [c_1 C(\varepsilon_a) - c_2 S(\varepsilon_a)]}{v [-C'(\varepsilon_a) S(\varepsilon_a) + C(\varepsilon_a) S'(\varepsilon_a)]} \right\} \right), \tag{17}$$

where $C(\varepsilon_a), S(\varepsilon_a), C'(\varepsilon_a)$ and $S'(\varepsilon_a)$ are the Mathieu S and Mathieu C functions and the first derivatives. For basic properties we refer to [31]. For a complex study about Mathieu functions see [34–36]. In (17), we used the abbreviation of $\varepsilon_a = -\frac{c^2}{v^2}, -\frac{a\lambda}{v^2}, -\frac{\pi}{4} + \frac{\omega}{2}$.

Figure 13 shows a typical shape function for the periodic noise term. Due to the elaborate properties of even the single Mathieu C or S functions for some parameter pairs a, q the function is finite with periodic oscillations and for some neighboring parameters it is divergent for large arguments. No general parameter dependence can be stated. The parameter space of the set of six real values $(c_1, c_2, c, a, v, \lambda)$ is too large to map. After the evaluation of numerous shape functions we may state, that a typical shape function is presented with two larger islands close to the origin and numerous smaller intervals. For large argument ω the shape function shows a steep decay.

Fig. 13 The shape function for the periodic noise. The applied parameters are $c_1 = 0.5, c_2 = 2, c = 1, \nu = 1, a = 1, \lambda = 3$

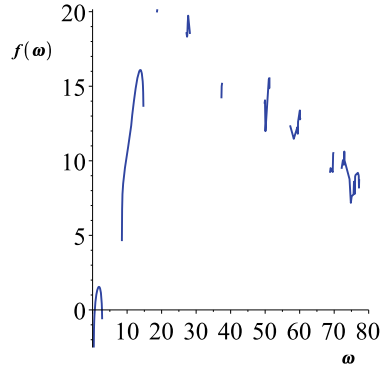


Fig. 14 The complete traveling wave solution $u(x, t)$ for periodic noise with the same parameter set as given above

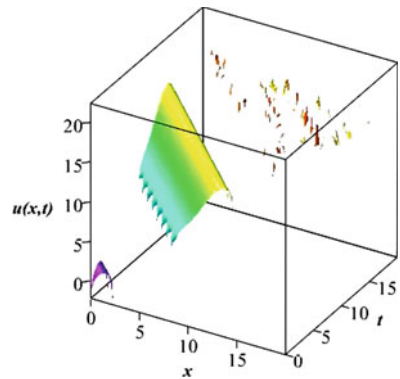


Figure 14 shows the complete solution. Note, that the first two broader islands can be seen as they freely travel. Due to the finite resolution the smaller islands are represented as irregular noise in the background.

5 Conclusions

In summary, we can say that with an appropriate change of variables applying the traveling-wave Ansatz one may obtain analytic solution for the KPZ equation for one spatial dimension with numerous noise terms. We investigated four type of power-law noise ω^n with exponents of $-2, -1, 0, 1$, called the brown, pink, white and blue noise, respectively. Each integer exponent describes completely different dynamics. Additionally, the properties of Gaussian and Lorentzian noises are investigated. Providing completely dissimilar surfaces with growth dynamics. All solutions can be described with non-trivial combinations of various special functions, like error, Whittaker, Kummer or Heun. The parameter dependencies of the solutions are investigated

and discussed. Future works are planned for the investigations of two dimensional surfaces.

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