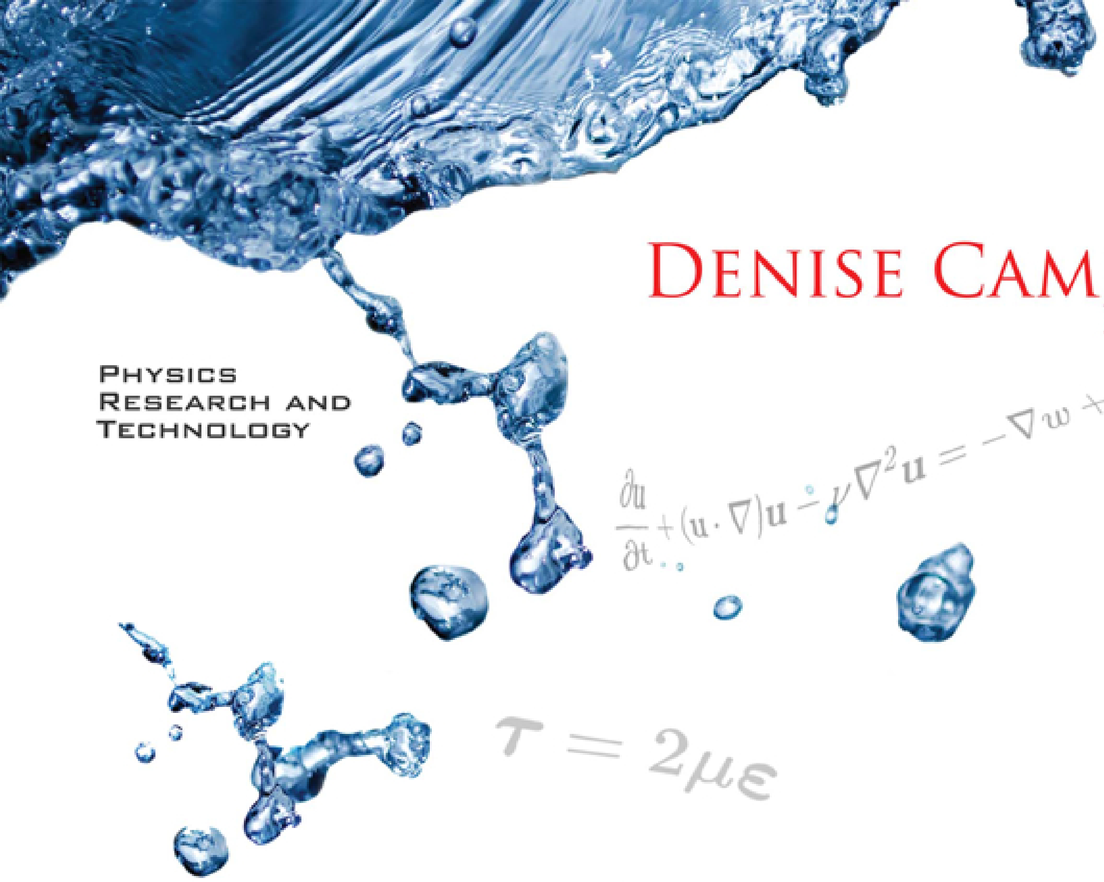

Denise Campos
Editor

Physics Research and Technology

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \mathbf{u} = -\nabla w + \mathbf{g}$$

$$\tau = 2\mu\varepsilon$$

$$\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} - \nu \frac{\partial^2}{\partial x_j \partial x_j}\right) u_i = -\frac{\partial w}{\partial x_i} + g_i$$

$$\tau = \mu\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$$

Handbook on Navier-Stokes Equations

Theory and Applied Analysis

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HANDBOOK ON NAVIER-STOKES EQUATIONS

THEORY AND APPLIED ANALYSIS

DENISE CAMPOS
EDITOR

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NOTICE TO THE READER

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CONTENTS

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Chapter 16

SELF-SIMILAR ANALYSIS OF VARIOUS NAVIER-STOKES EQUATIONS IN TWO OR THREE DIMENSIONS

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Abstract

In the following chapter we will shortly introduce the self-similar Ansatz as a powerful tool to attack various non-linear partial differential equations and find - physically relevant - dispersive solutions. Later, we classify the Navier-Stokes (NS) equations into four subsets, like Newtonian, non-Newtonian, compressible and incompressible. This classification is arbitrary, however helps us to get an overview about the structure of various viscous fluid equations. We present our analytic solutions for three of these classes. The relevance of the solutions are emphasized. Lastly, we present an interesting system the Oberbeck-Boussinesq system where the two dimensional NS equation is coupled to heat conduction. This system pioneered the way to chaos studies about half a century ago. Of course, the self-similar solution is presented which helps us to enlighten the formation of the Rayleigh-Bénard convection cells.

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1. Introduction

There are no existing methods which could help us to solve non-linear partial differential equations (PDE) in general. However, two basic linear and time-dependent PDE exist which could help us. The first is the hyperbolic second order wave equation, in one dimension the well-known form is $u_{xx} - \frac{1}{c^2}u_{tt}$ where the subscripts mean the partial derivation, c is the wave propagation velocity (always a finite value), and u is the physical quantity which propagates. Traveling waves - in both directions - are the general solutions of this problem with the form of $u(x \pm ct)$. This is a more-or-less common knowledge in the theoretical physics community. A detailed study where traveling waves are used to solve various PDEs is [1].

The other most important linear PDE is the diffusion or heat conduction equation. In one dimension it reads $u_t = au_{xx}$ where a is the diffusion or heat conduction coefficient should be a positive real number. The well-known solution is the Gaussian curve which describes the decay and spreading of the initial particle or heat distribution. There is a rigorous mathematical theorem - the strong maximum principle - which states that the solution of the diffusion equation is bounded from above. The problem of this equation is the infinite signal propagation speed, which basically means that the Gaussian curve has no compact support, but that is not a relevant question now. The main point - which is the major motivation of this chapter - is that there exist a natural Ansatz (or trial function) which solves this equation. Namely, the self-similar solution, from basic textbooks the one-dimensional form is well-known [2, 3, 4]

$$T(x,t) = t^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) := t^{-\alpha} f(\eta), \tag{1}$$

where $T(x,t)$ can be an arbitrary variable of a PDE and t means time and x means spatial dependence. The similarity exponents α and β are of primary physical importance since α gives the rate of decay of the magnitude $T(x,t)$, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution as time goes on. The most powerful result of this Ansatz is the fundamental or Gaussian solution of the Fourier heat conduction equation (or for Fick's diffusion equation) with $\alpha = \beta = 1/2$. These solutions are visualized in Figure 1. for different time-points $t_1 < t_2$. Applicability of this Ansatz is quite wide and comes up in various transport systems [2, 3, 4, 5, 6, 7]. Solutions with integer exponents are called self-similar solutions of the first kind (and sometimes can be obtained from dimensional analysis of the problem as well). The above given Ansatz can be generalized considering real and continuous functions a(t) and b(t) instead of t^{α} and t^{β}. This transformation is based on the assumption that a self-similar solution exists, i.e., every physical parameter preserves its shape during the expansion. Self-similar solutions usually describe the asymptotic behavior of an unbounded or a far-field problem; the time t and the space coordinate x appear only in the combination of $f(x/t^{\beta})$. It means that the existence of self-similar variables implies the lack of characteristic length and time scales. These solutions are usually not unique and do not take into account the initial stage of the physical expansion process. These kind of solutions describe the intermediate asymptotic of a problem: they hold when the precise initial conditions are no longer important, but before the system has reached its final steady state. For some systems it can be shown that the self-similar solution fulfills the source type (Dirac delta) initial condition, but not in every case. These Ansätze are much simpler than the full solutions of the PDE and so easier to understand and study in different regions of parameter space. A final reason for studying them is that they are solutions of a system of ordinary differential equations and hence do not suffer the extra inherent numerical problems of the full partial differential equations. In some cases self-similar solutions helps to understand diffusion-like properties or the existence of compact supports of the solution.

Finally, it is important to emphasize that the self-similar Ansatz has an important but not well-known and not rigorous connection to phase transitions and critical phenomena. Namely to scaling, universality and renormalization. As far as we know even genuine pioneers of critical phenomena like Stanley [8] cannot undertake to give a rigorous clear-cut definitions for all these conceptions, but we fell that all have a common root. The starting point could be the generalized homogeneous function like the Gibbs potential $G_s(H,\epsilon)$ for a spin system. Close to the critical point the scaling hypothesis can be expressed via the following mathematical rule $G_s(\lambda^a H, \lambda^b \epsilon) = \lambda G_s(H,\epsilon)$. Where H is the order parameter the magnetic field and ϵ is the reduced temperature, a, b are the critical exponents. The same exponents mean the same universality classes. The equation gives the definition of homogeneous functions. Empirically, one finds that all systems in nature belong to one of a comparatively small number of such universality classes. The scaling hypothesis predicts that all the curves of this family $M(H,\epsilon)$ can be "collapse" onto a single curve provided one plots not M versus ϵ but rather a scaled M (M divided by H to some power) vs a scaled ϵ (ϵ divided by H to some different power). The renormalization approach to critical phenomena leads to scaling. In renormalization the exponent is called the scaling exponent. We hope that this small turn-out helps th reader to a much better understanding of our approach.

After this successful physical interpretation of this solution we may try to apply it to any kind of PDE system which has some dissipative property (all the NS equations are so) and look what kind of results we get.

Unfortunately, there is no direct analytic calculation with the 3 dimensional self-similar generalization of this Ansatz in the literature. Now we show how this generalization is possible and what does it geometrically means.

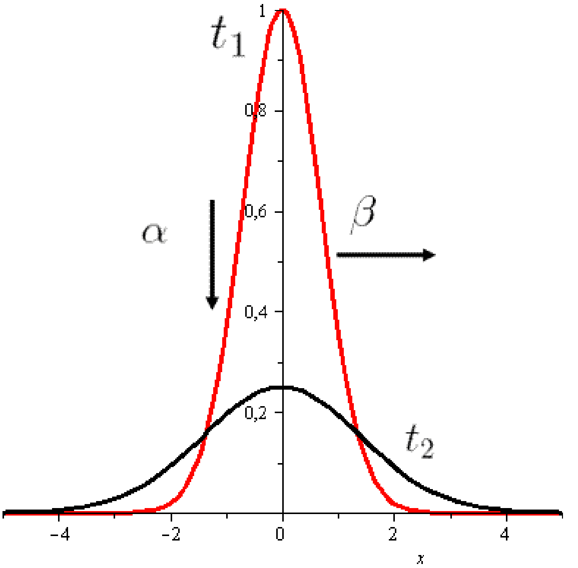

Figure 1. A self-similar solution of Eq. (1) for $t_1 < t_2$. The presented curves are Gaussians for regular heat conduction.

This Ansatz can be generalized for two or three dimensions in various ways one is the following

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{F(x,y,z)}{t^{\beta}}\right) := t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right) := t^{-\alpha} f(\omega) \qquad (2)$$

where $F(x,y,z)$ can be understood as an implicit parameterization of a two dimensional surface. If the function $F(x,y,z) = x + y + z = 0$ which is presented in Figure 2. then it is an implicit form of a plane in three dimensions. At this point we can give a geometrical interpretation of the Ansatz. Note that the dimension of $F(x,y,z)$ still have to be a spatial coordinate. Taking the following incompressible NS equation as an example, with this Ansatz we consider all the x coordinate of the velocity field $\mathbf{v_x} = u$ where the sum of the spatial coordinates are on a plane on the same footing. We are not considering all the R^3 velocity field but a plane of the $\mathbf{v}_x$ coordinates as an independent variable. The NS equation - which is responsible for the dynamics - maps this kind of velocities which are on a surface to another geometry. In this sense we can investigate the dynamical properties of the NS equation truly. In principle there are more possible generalization of the Ansatz available.

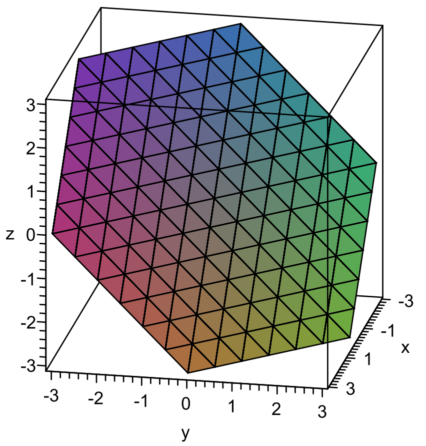

Figure 2. The graph of the $x + y + z = 0$ plane.

One is the following:

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{\sqrt{x^2+y^2+z^2-a}}{t^{\beta}}\right) := t^{-\alpha} f(\omega) \tag{3}$$

which can be interpreted as an Euclidean vector norm or L^2 norm. Now we contract all the x coordinate of the velocity field u (which are on a surface of a sphere with radius a) to a simple spatial coordinate. Unfortunately, if we consider the first and second spatial derivatives and plug them into the NS equation we cannot get a pure η dependent ordinary differential equation(ODE) system some explicit x, y, z or t dependence tenaciously remain. For a telegraph-type heat conduction equation (where is no $\mathbf{v}\nabla\mathbf{v}term)$ both of these Ansätze are useful to get solutions for the two dimensional case [7].

Lastly, in the introduction we present a table which classifies the viscous fluid equations - this alignment is arbitrary - however it gives a better and more compact view into the fascinating world of NS equations. The three fields which are marked with 'X' will be analyzed in details in the following. The fourth group of NS equations is out of our recent

Table 1. A classification of various NS equations

type	Newtonian	non-Newtonian
incompressible	X	X
compressible	X	-

scope. At first we are not sure how to define it in a consistent way. Secondly, this system would be far too complex to analyse with the self-similar trial function.

2. Incompressible Newtonian Fluids

In Cartesian coordinates and Eulerian description the NS and the continuity equations are the following:

$$\begin{aligned} \nabla \mathbf{v} &= 0, \\ \mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} &= \nu \triangle \mathbf{v} - \frac{\nabla p}{\rho} + a \end{aligned} \tag{4}$$

where $\mathbf{v}, \rho, p, \nu, a$ denote respectively the three-dimensional velocity field, density, pressure, kinematic viscosity and an external force (like gravitation). (To avoid further misunderstanding we use a for external field instead of the letter g which is reserved for a self-similar shape function.) From now on ρ, ν, a are physical parameters of the flow. For a better transparency we use the coordinate notation for the velocity $\mathbf{v}(x,y,z,t) = u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)$ and for the scalar pressure variable $p(x,y,z,t)$

$$\begin{aligned} u_x + v_y + w_z &= 0, \\ u_t + uu_x + vu_y + wu_z &= \nu(u_{xx} + u_{yy} + u_{zz}) - \frac{p_x}{\rho}, \\ v_t + uv_x + vv_y + wv_z &= \nu(v_{xx} + v_{yy} + v_{zz}) - \frac{p_y}{\rho}, \\ w_t + uw_x + vw_y + ww_z &= \nu(w_{xx} + w_{yy} + w_{zz}) - \frac{p_z}{\rho} + a. \end{aligned} \tag{5}$$

The subscripts mean partial derivations. According to our best knowledge there are no analytic solution for the three dimensional most general case. However, there are numerous examination techniques available in the literature. Manwai [9] studied the N-dimensional $(N \geq 1)$ radial Navier-Stokes equation with different kind of viscosity and pressure dependences and presented analytical blow up solutions. His works are still 1+1 dimensional (one spatial and one time dimension) investigations. Later, in a book the stability and blow up phenomena of various isentropic Euler-Poisson, Navier-Stokes-Poisson, Navier-Stokes, Euler equations were examined [10]. Another popular and well established investigation method is based on Lie algebra there are numerous studies available. Some of them are even for the three dimensional case, for more details see [11]. Unfortunately, no explicit solutions are shown and analyzed there. Fushchich et al. [12] constructed a complete set of $\tilde{G}(1,3)$-inequivalent Ansätze of co-dimension one for the NS system, they present nineteen different analytical solutions for one or two space dimensions. They last solution is very closed to our one but not identical, we will come back to these results later. Further two and three dimensional studies based on other group analytical method were presented by Grassi [13]. They also present solutions which look almost the same as ours, but they consider only two space dimensions. We will compare these results to our one at the end of this section.

Some years ago, Hu et al. [14] presents a study where symmetry reductions and exact solutions of the (2+1)-dimensional NS were presented. Aristov and Polyanin [15] use various methods like generalized separation of variables, Crocco transformation or the method of functional separation of variables for the NS and present large number of new classes of exact solutions. Sedov in his classical work [2] (Page 120) presented analytic solutions for the tree dimensional spherical NS equation where all three velocity components and the pressure have polar angle dependence (θ) only. Even this restricted symmetry led to a non-linear coupled ordinary differential equation system with has a very nice mathematical structure.

Now we concentrate on the first Ansatz (2) and search the solution of the NS PDE system in the following form:

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right), \quad v(x,y,z,t) = t^{-\gamma} g\left(\frac{x+y+z}{t^{\delta}}\right),$$
$$w(x,y,z,t) = t^{-\epsilon} h\left(\frac{x+y+z}{t^{\zeta}}\right), \quad p(x,y,z,t) = t^{-\eta} l\left(\frac{x+y+z}{t^{\theta}}\right). \tag{6}$$

Where all the exponents $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta$ are real numbers. (Solutions with integer exponents are called self-similar solutions of the first kind and non-integer exponents mean self-similar solutions of the second kind.) The functions f, g, h, l are arbitrary and will be evaluated later on. According to the NS system we need to calculate all the first time derivatives of the velocity field, all the first and second spatial derivatives of the velocity fields and the first spatial derivatives of the pressure. We skip these trivial calculations here. Note that both Eq. (5) and Eq. (6) have a large degree of exchange symmetry in the coordinates x, y and z. We want to get an ODE system for all the four functions $f(\omega), g(\omega), h(\omega), l(\omega)$ which all have to have the same argument ω. This dictates the constraint that $\beta = \delta = \zeta = \theta$ have to be the same real number which reduces the number of free parameters, (let's use the β from now on $\omega = \frac{x+y+z}{t^{\beta}}$). From this constrain follows that e.q. $u_x = \frac{f'(\omega)}{t^{\alpha+\beta}} \approx v_y = \frac{f'(\omega)}{t^{\gamma+\beta}}$ where prime means derivation with respect to ω. This example clearly shows the hidden symmetry of this construction which may helps us. For the better transparency we present the second equation of (5) after the substitution of the Ansatz (6)

$$-\alpha t^{-\alpha-1} f(\omega) - \beta t^{-\alpha-1} f'(\omega)\omega + t^{-2\alpha-\beta} f(\omega) f'(\omega) + t^{-\gamma-\alpha-\beta} g(\omega) f'(\omega) +$$
$$t^{-\epsilon-\alpha-\beta} h(\omega) f'(\omega) = \nu 3 t^{-\alpha-2\beta} f''(\omega) - \frac{t^{-\mu-\beta} l'(\omega)}{\rho}. \tag{7}$$

To have an ODE which only depends on ω (which is now the new variable instead of time t and the radial components) all the time dependences e.g. $t^{-\alpha-1}$ have to be zero OR all the exponents have to have the same numerical value. After some algebraic manipulation it becomes clear that all the six exponents $\alpha - \zeta$ included for the velocity field (the first three functions in Eq. (6)) have to be $+1/2$. The only exception is the term with the gradient of the pressure. There $\eta = 1$ and $\theta = 1/2$ have to be. Now in Eq. (7) each term is multiplied by $t^{-3/2}$. Self-similar exponents with the value of $+1/2$ are well-known from the regular Fourier heat conduction (or for the Fick's diffusion) equation and gives back the fundamental solution which is the usual Gaussian function. This is a fundamental knowledge, that NS is a kind of diffusion equation for the velocity field. For pressure the $\eta = 1$ exponent means, a two times quicker decay rate of the magnitude than for the velocity field.

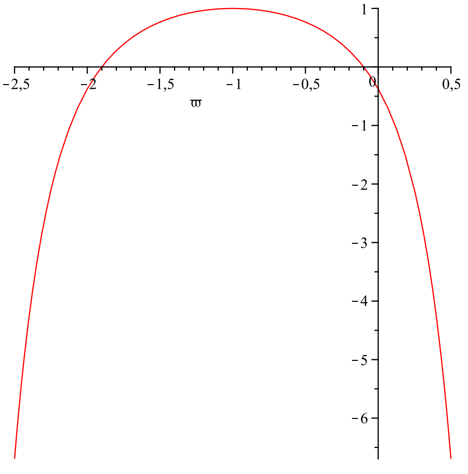

Figure 3. The $KummerM\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right)$ function for $c = 1$, and $\nu = 0.1$.

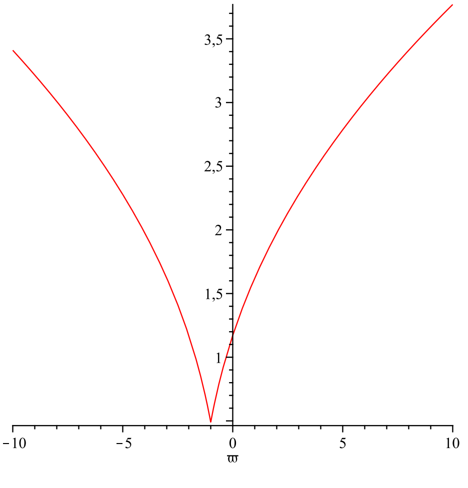

Figure 4. The $KummerU\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right)$ function for $c = 1$, and $\nu = 0.1$.

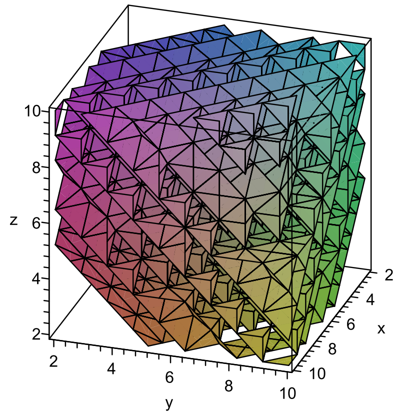

Figure 5. The implicit plot of the self-similar solution Eq. (15). Only the KummerU function is presented for $t = 1, c_1 = 1, c_2 = 0, a = 0, c = 1$, and $\nu = 0.1$.

The corresponding coupled ODE system is

$$\begin{aligned} f'(\omega) + g'(\omega) + h'(\omega) &= 0, \\ -\frac{1}{2}f(\omega) - \frac{1}{2}\omega f'(\omega) + [f(\omega)+g(\omega)+h(\omega)]f'(\omega) &= 3\nu f''(\omega) - \frac{l'(\omega)}{\rho}, \\ -\frac{1}{2}g(\omega) - \frac{1}{2}\omega g'(\omega) + [f(\omega)+g(\omega)+h(\omega)]g'(\omega) &= 3\nu g''(\omega) - \frac{l'(\omega)}{\rho}, \\ -\frac{1}{2}h(\omega) - \frac{1}{2}\omega h'(\omega) + [f(\omega)+g(\omega)+h(\omega)]h'(\omega) &= 3\nu h''(\omega) - \frac{l'(\omega)}{\rho} + a. \end{aligned} \quad (8)$$

From the first (continuity) equation we automatically get

$$f(\omega) + g(\omega) + h(\omega) = c, \quad and \quad f''(\omega) + g''(\omega) + h''(\omega) = 0 \quad (9)$$

where c is proportional with the constant mass flow rate. Implicitly, larger c means larger velocities. After some algebraic manipulation of all the three NS equations we get the final equation

$$9\nu f''(\omega) - 3(\omega + c)f'(\omega) + \frac{3}{2}f(\omega) - \frac{c}{2} + a = 0. \quad (10)$$

The solutions are the Kummer functions [16]

$$f(\omega) = c_1 \cdot KummerU\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right) + c_2 \cdot KummerM\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3} \quad (11)$$

where c_1 and c_2 are integration constants. The KummerM function is defined by the following series

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + ... + \frac{(a)_n z^n}{(b)_n n!} \quad (12)$$

where $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)...(a+n-1), (a)_0 = 1. \quad (13)$$

The KummerU function can be defined from the KummerM function via the following form

$$U(a,b,z) = \frac{\pi}{sin(\pi b)}\left[\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right] \quad (14)$$

where $\Gamma()$ is the Gamma function. Exhausted mathematical properties of the Kummer function can be found in [16].

Note, that the solution depends only on two parameters where the ν is the viscosity, and c is proportional with the mass flow rate. Figure 3 and Figure 4 show the KummerM and KummerU function for $c = 1$ and $\nu = 0.1$, respectively. For stability analysis we note that the power series which is applied to calculate the Kummer functions has a pure convergence and a 30 digit accuracy was needed to plot the KummerU function, otherwise spurious oscillations occurred on the figure. (Here we note, that the Malpe 12 Software was used during our analysis.) Note, that for $\omega = 6.5$ the KummerM goes to infinity, and $\omega \to \infty$ KummerU function goes to ∞ which is physically hard to understand, which means that the velocity field goes to infinity as well.

The complete self-similar solution of the x coordinate of the velocity reads

$$\begin{aligned} u(x,y,z,t) = \quad & t^{-1/2} f(\omega) = t^{-1/2}\left[c_1 \cdot KummerU\left(\frac{-1}{4},\frac{1}{2},\frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\right)\right] + \\ & t^{-1/2}\left[c_2 \cdot KummerM\left(-\frac{1}{4},\frac{1}{2},\frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}\right]. \end{aligned} \quad (15)$$

In Figure 5 an implicit plot of Eq. (15) is visualized. The KummerU function was presented only, the used parameters are the following $c_1 = 1, c_2 = 0, t = 1, c = 1, \nu = 0.1, a = 0$. Note, that the initial flat surface of Figure 2 is mapped into a complicated topological surface via the NS dynamical equation. The following phenomena happened, an implicit function is presented, we already mentioned that all the $x + y + z = 0$ points considered to be the same. Therefore we got a multi-valued surface because for a fixed x numerical value various y+z combinations give the same argument inside the Kummer functions. Unfortunately, this effect is hard to visualize. This can be understand as a kind of fingerprint of a turbulence-like phenomena which is still remained in the equation. An initial simple single-valued plane surface is mapped into a very complicated multivalued surface. Note, that for a larger value (now we presented KummerU() = 2 case) or for larger flow rate (c=1) the surface got even more structure. Therefore, Figure 5 presents only a principle. At this point we can also give statements about the stability of this solution, the solution the Kummer functions are fine, but for larger flow values a more precise and precise calculation of the solution surface is needed which means larger computational effort which is well known from the application of the NS equation.

From the integrated continuity equation $(f = c-g-f)$ we automatically get an implicit formula for the other two velocity components

$$\begin{aligned} v(x,y,z,t) + \quad w \quad (x,y,z,t) = & -t^{-1/2}\left[c_1 \cdot KummerU\left(\frac{-1}{4},\frac{1}{2},\frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\right)\right] - \\ & t^{-1/2}\left[c_2 \cdot KummerM\left(-\frac{1}{4},\frac{1}{2},\frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}\right] + c. \end{aligned} \quad (16)$$

For explicit formulas of the next velocity component the following ODE has to integrated

$$-3\nu g''(\omega) + g'(\omega)\left(-\frac{\omega}{2} + c\right) - \frac{g(\omega)}{2} + F(f''(\omega), f'(\omega), f(\omega)) = 0 \quad (17)$$

where $F(f''(\omega), f'(\omega), f(\omega))$ contains the combination of the first and second derivatives of the Kummer functions. This is a second order linear ODE and the solution can be obtained with the following general quadrature

$$g(\omega) = \left[c_2 + \int \left\{ \frac{-c_1 + \int F(f''(\omega), f'(\omega), f(\omega))d\omega \cdot exp\left(\frac{-\omega^2/4+c\omega}{-3\nu}\right)}{3\nu} \right\} d\omega \right] \times \quad (18)$$

$$exp\left(\frac{-\omega^2/4 + c\omega}{3\nu}\right). \quad (19)$$

There are additional recursion formulas which help to express the first and second derivatives of the KummerU functions. Such technical details can be found in the original publication [17]. Unfortunately, we could not find any closed form for $v(x,y,z,t)$ and for $w(x,y,z,t)$. Only v the x coordinate of the velocity $\mathbf{v}$ field can be evaluated in a closed form in this manner. To overcome this problem, we may call symmetry considerations. The continuity equation states that the sum of all the three velocity component should give a constant. Therefore if every velocity component has the form of (11), we get a solution.

As we mentioned at the beginning there are analytic solutions available in the literature which are very similar to our one. Fushchich et al. [12] present 19 different solutions for the full three dimensional NS and continuity equation. (For a better understanding we used the same notation here as well.) For the last (19.) solution they apply the following Ansatz of

$$u(z,t) = \frac{f(\omega)}{\sqrt{t}}, \;\; v(y,z) = \frac{g(\omega)}{\sqrt{t}} + \frac{y}{t}, \;\; w(z,t) = \frac{h(\omega)}{\sqrt{t}}, \;\; p(t,z) = \frac{l(\omega)}{\sqrt{t}} \quad (20)$$

where $\omega = z/\sqrt{t}$ is the invariant variable. The obtained ODE is very similar to ours (8) .

$$\begin{aligned} h'(\omega) + 1 &= 0 \\ -\frac{1}{2}(f(\omega) + \omega f'(\omega)) + h(\omega) f'(\omega) &= f''(\omega), \\ \frac{1}{2}(g(\omega) + \omega g'(\omega)) + h(\omega) g'(\omega) &= g''(\omega), \\ -\frac{1}{2}(h(\omega) + \omega h'(\omega)) + h(\omega) h'(\omega) + l'(\omega) &= f''(\omega). \end{aligned} \quad (21)$$

The solutions are

$$\begin{aligned} f(\omega) &= (\frac{3}{2}\omega - c)^{-1/2} exp\left[-\frac{1}{6}(\frac{3}{2}\omega - c)^2\right] w\left[-\frac{1}{12}, \frac{1}{4}, \frac{1}{3}(\frac{3}{2}\omega - c)^2\right] \\ g(\omega) &= (\frac{3}{2}\omega - c)^{-1/2} exp\left[-\frac{1}{6}(\frac{3}{2}\omega - c)^2\right] w\left[-\frac{5}{12}, \frac{1}{4}, \frac{1}{3}(\frac{3}{2}\omega - c)^2\right] \\ h(\omega) &= -\omega + c \\ l(\omega) &= \frac{3}{2}c\omega - \omega^2 + c_1 \end{aligned} \quad (22)$$

where w is the Whittaker function, c and c_1 are integration constants. Note that the Whittaker and the Kummer functions are strongly related to each other [16]

$$w(\kappa, \mu, z) = e^{-1/2z} z^{1/2+\mu} KummerM(1/2 + \mu - \kappa, 1 + 2\mu, z). \quad (23)$$

More details can be found in the original work [12].

As a second comparison we show the results of [13]. They also have a modified form of (29) which is the following

$$\begin{aligned} U_{1t} + cU_1 + U_2U_{1y} + U_3U_{1z} - \nu(U_{1yy} + U_{1zz}) &= 0, \\ U_{2t} + U_2U_{2y} + U_3U_{2z} + \pi_y - \nu(U_{2yy} + U_{2zz}) &= 0, \\ U_{3t} + U_2U_{3y} + U_3U_{3z} + \pi_z - \nu(U_{3yy} + U_{3zz}) &= 0, \\ U_{2y} + U_{3z} + c &= 0 \end{aligned} \tag{24}$$

where $U_i, i = 1..3$ are the velocity components $U_i(y, z, t)$ and π is the pressure, c stands for constants, ν is viscosity and additional subscripts mean derivations. After some transformation they get a linear PDA as follows

$$U_{1t} + k_1yU_{1y} + (\sigma - k_1z)U_{1z} - \nu(U_{1yy} + U_{1zz}) = 0 \tag{25}$$

it is convenient to look the solution in the form of

$$U_1 = Y(y)T(z)\Phi(t). \tag{26}$$

Note, that they also consider the full 3 dimensional problem, but the velocity filed has a restricted two dimensional(y,z) coordinate dependence. There are additional conditions but the general solution can be presented

$$\begin{aligned} \Phi &= c_1 exp(c_2)t \\ Y &= c_3M\left(-c_4, \frac{1}{2}, \frac{y^2}{2\nu}\right) + yc_5M\left(\frac{1}{2} - c_4, \frac{3}{2}, \frac{y^2}{2\nu}\right) \\ T &\approx M\left(c_6, \frac{1}{2}, \frac{z^2}{2\nu}\right) + zM\left(\frac{1}{2} - c_6, \frac{3}{2}, \frac{z^2}{2\nu}\right) \end{aligned} \tag{27}$$

where M is the KummerM function as was presented below. The exact solution in [13] (Eq. 4.10a-4.10c) contains more constants as presented here. It is not our goal to reproduce the full calculation of [13] (which is not our work) we just want to give a guideline to their solution vigorously emphasizing that our solution is very similar to the presented one. Note that in both results the arguments of the KummerM function (11) and (27) are proportional to the square of the radial component divided by the viscosity, additionally one of the parameters is 1/2. As a last word we just would like to say, (as this example clearly shows) that the Lie algebra method is not the exhaustive method to find all the possible solutions of a PDA.

It is possible that even this moderate result can give any simulating impetus to the numerical investigation of the NS equation. Our solution can be used as a test case for various numerical methods or commercial computer packages like Fluent or CFX.

3. Compressible Newtonian Fluids

To study the dynamics of viscous compressible fluids the compressible NS together with the continuity equation have to be investigated. In Eulerian description in Cartesian coordinates these equations are the following:

$$\begin{aligned} \rho_t + \mathrm{div}[\rho \mathbf{v}] &= 0 \\ \rho[\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v}] &= \nu_1 \triangle \mathbf{v} + \frac{\nu_2}{3}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v} - \nabla p + a \end{aligned} \tag{28}$$

where $\mathbf{v}, \rho, p, \nu_{1,2}$ and a denote respectively the three-dimensional velocity field, density, pressure, kinematic viscosities and an external force (like gravitation) of the investigated fluid. As before we use a for external field instead of the letter g which is reserved for a self-similar solution. In the later we consider no external force, so $a = 0$. For physical completeness we need an equation of state (EOS) to close the equations. We start with the polytropic EOS $p = \kappa \rho^n$, where κ is a constant of proportionality to fix the dimension and n is a free real parameter (n is usually less than 2). In astrophysics, the Lane - Emden equation is a dimensionless form of the Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid. It's solution is the polytropic EOS which we apply in the following. The question of more complex EOSs will be concerned later. Now $\nu_{1,2}, a, \kappa, n$ are parameters of the flow. To have a better overview we use the coordinate notation $\mathbf{v}(x,y,z,t) = (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$ and for the scalar density variable $\rho(x,y,z,t)$ from now on. Having in mind the correct forms of the mentioned complicated vector operations, the PDE system reads the following:

$$\begin{aligned} \rho_t + \rho_x u + \rho_y v + \rho_z w + \rho[u_x + v_y + w_z] = 0, \\ \rho[u_t + uu_x + vu_y + wu_z] - \nu_1[u_{xx} + u_{yy} + u_{zz}] - \frac{\nu_2}{3}[u_{xx} + v_{xy} + w_{xz}] + \kappa n \rho^{n-1}\rho_x = 0, \\ \rho[v_t + uv_x + vv_y + wv_z] - \nu_1[v_{xx} + v_{yy} + v_{zz}] - \frac{\nu_2}{3}[u_{xy} + v_{yy} + w_{yz}] + \kappa n \rho^{n-1}\rho_y = 0, \\ \rho[w_t + uw_x + vw_y + ww_z] - \nu_1[w_{xx} + w_{yy} + w_{zz}] - \frac{\nu_2}{3}[u_{xz} + v_{yz} + w_{zz}] + \kappa n \rho^{n-1}\rho_z = 0. \end{aligned} \tag{29}$$

The subscripts mean partial derivations as was defined earlier. Note, that the formula for EOS is already applied.

There is no final and clear-cut existence and uniqueness theorem for the most general non-compressible NS equation. However, large number of studies deal with the question of existence and uniqueness theorem related to various viscous flow problems. Without completeness we mention two works which (together with the references) give a transparent overview about this field [18, 19].

According to our best knowledge there are no analytic solutions for the most general three dimensional NS system even for non-compressible Newtonian fluids. Additional similarity reduction studies are available from various authors as well [20, 21, 22]. A full three dimensional Lie group analysis is available for the three dimensional Euler equation of gas dynamics, with polytropic EOS [23] unfortunately without any kind of viscosity.

We use the above mentioned three dimensional Ansatz again. It can be easily shown than even a more general plane, like $ax + by + dz + 1 = 0$ makes the remaining ODE system much more complicated. (The second term in the NS equation on the right hand side (grad div $\mathbf{v}$ term) creates distinct a^2, b^2, c^2 terms which cannot be transformed out, and a coupled system of three equations remain.)

The Ansatz we apply is:

$$\rho(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right) = t^{-\alpha} f(\eta), \quad u(\eta) = t^{-\delta} g(\eta),$$
$$v(\eta) = t^{-\epsilon} h(\eta), \quad w(\eta) = t^{-\omega} i(\eta), \tag{30}$$

where all the exponents $\alpha, \beta, \delta, \epsilon, \omega$ are real numbers as usual. The technical part of the calculation is analogous as was shown above. To get a final ODE system which depends only on the variable η, the following universality relations have to be hold

$$\alpha = \beta = \frac{2}{n+1} \qquad \& \qquad \delta = \epsilon = \omega = \frac{2n-2}{n+1}, \tag{31}$$

where $n \neq -1$. Note, that the self-similarity exponents are not fixed values thanks to the existence of the polytropic EOS exponent n. (In other systems e.g. heat conduction or non-compressible NS system, all the exponents have a fixed value, usually +1/2.) This means that our self-similar Ansatz is valid for different kind of materials with different kind of EOS. Different exponents represent different materials with different physical properties which results different final ODEs with diverse mathematical properties.

At this point we have to mention that a NS equation even with a more complicated EOS like $p \sim f(\rho^l v^m)$ could have self-similar solutions. We may say in general, that EOSs obtained from Taylor series expansion taking into account many term are problematic and give contradictory relations among the exponents. The investigation of such problems will be performed in the near future but not in the recent study.

Our goal is to analyze the asymptotic properties of Eq. (30) with the help of Eq. (31). According to Eq. (1) the signs of the exponents automatically dictates the asymptotic behavior of the solution at sufficiently large time. All physical velocity components should decay at large times for a viscous fluid without external energy source term. The role of α and β was explained after Eq. (1). Figure 6 shows the $\alpha(n)$ and $\delta(n)$ functions. There are five different regimes:

- $n > 1$ all exponents are positive - physically fully meaningful scenario - spreading and decaying density and all speed components for large time - will be analyzed in details for general n
- $n = 1$ spreading and decaying density in time and spreading but non-decaying velocity field in time - not completely physical but the simplest mathematical case
- $-1 \leq n \leq 1$ spreading and decaying density in time and and spreading and enhancing velocity in time - not a physical scenario
- $n \neq -1$ not allowed case
- $n \leq -1$ sharpening and enhancing density and sharpening and decaying velocity in time, we consider it an non-physical scene and neglect further analysis.

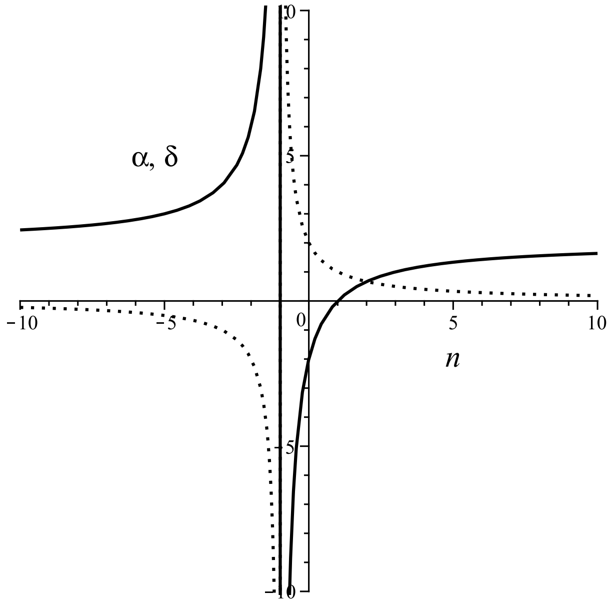

Figure 6. Eq. (31) dotted line is $\alpha(n) = 2/(n+1)$ and solid line is $\delta(n) = 2 - 4/(n+1)$.

The corresponding coupled ODE system is:

$$\begin{aligned}
\alpha[f + f'\eta] &= f'[g+h+i] + f[g'+h'+i'], \\
f[-\delta g - \alpha\eta g' + gg' + hg' + ig'] &= -\kappa n f^{n-1} f' + 3\nu_1 g'' + \frac{\nu_2}{3}[g''+h''+i''], \\
f[-\delta h - \alpha\eta h' + gh' + hh' + ih'] &= -\kappa n f^{n-1} f' + 3\nu_1 h'' + \frac{\nu_2}{3}[g''+h''+i''], \\
f[-\delta i - \alpha\eta i' + gi' + hi' + ii'] &= -\kappa n f^{n-1} f' + 3\nu_1 i'' + \frac{\nu_2}{3}[g''+h''+i''],
\end{aligned} \tag{32}$$

where prime means derivation with respect to η. The continuity equation is a total derivative if $\alpha = \beta$, therefore we can integrate getting $\alpha f \eta = f[g+h+i]+c_0$, where c_0 is proportional to the mass flow rate. For the shake of clarity, we simplify the NS equation with introducing a single viscosity $\nu = \nu_1 = \nu_2$. There are still too many free parameters remain for the general investigation.

We fix $c_0 = 0$. Having in mind that the density of a fluid should be positive $f \neq 0$, we get $\alpha\eta = g + h + i$. With the help of the first and second derivatives of this formula Eq. (32) can be reduced to the next non-linear first order ODE

$$-3\kappa n f^{n-1} f' + \left(\frac{4n-4}{(n+1)^2}\right)\eta f = 0. \tag{33}$$

Note, that it is a first order equation, so there is a conserved quantity which should be a kind of general impulse in the parameter space η. Note, that this equation has no contribution from the viscous terms with ν just from the pressure and from the convective terms. The general solution can be derived with quadrature

$$f(\eta) = 3^{\frac{-1}{n-1}} \left(\frac{2\eta^2[n-1]}{\kappa n[n+1]} + 3c_1 \right).^{\frac{1}{n-1}} \tag{34}$$

Note, that for $\{n; n \in \mathbf{Z}\backslash\{-1\}\}$ exists n different solutions for $n > 0$ (one of them is the $f(\eta) = 0$) and $n-1$ different solutions for $n < 0$ these are the n or (n-1)th roots of the expression. For $\{n : n \in \mathbf{R}\backslash\{-1\}\}$ there is one real solution. It is remarkable, that for fixed $\kappa, c_1 > 0$ when n and η tend to infinity, the limit of Eq. (34) tends to zero. This meets our physical intuition for a viscous flow, we get back solutions which have an asymptotic decay. (In the limiting case $n = 1$ (which means the $\delta = 0$) we get back the trivial result $f = const.$ which is irrelevant.) For the $n = 2$, the least radical case $f(\eta) = \eta^2/(27\kappa) + c_1$ which is a quadratic function in η however, the full density function $\rho = t^{-2/3}[(x+y+z)^2/t^{4/3}] = (x+y+z)^2/t^2$ has a proper time decay for large times. This is consistent with our physical picture.

All the three velocity field components can be derived independently from the last three Eqs. (32). For the $v = t^{-\delta}g(\eta)$ the ODE reads:

$$-3\nu g'' + \left(\frac{2n-2}{n+1}\right) gf - \kappa n f^{n-1} f' = 0. \tag{35}$$

Unfortunately, there is no solution for general n in a closed form. However, for $n = 2$ the solutions can be given inserting $f(\eta) = \eta^2/(27\kappa)$ into Eq. (35). These are the Whittaker W and Whittaker M functions [16]

$$g = \frac{\tilde{c}_1}{\sqrt{\eta}} \mathrm{M}_{-\frac{c_1\sqrt{2\kappa}}{4\sqrt{\nu}}, \frac{1}{4}} \left(\frac{\sqrt{2}\eta^2}{9\sqrt{\nu\kappa}} \right) + \frac{\tilde{c}_2}{\sqrt{\eta}} \mathrm{W}_{-\frac{c_1\sqrt{2\kappa}}{4\sqrt{\nu}}, \frac{1}{4}} \left(\frac{\sqrt{2}\eta^2}{9\sqrt{\nu\kappa}} \right) + \frac{2}{3}\eta, \tag{36}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are integration constants. The M is the irregular and the W is the regular Whittaker function, respectively. These functions can be expressed with the help of the Kummer's confluent hypergeometric functions M and U in general (for details see [16])

$$\begin{aligned} M_{\lambda,\mu}(z) &= e^{-z/2} z^{\mu+1/2} M(\mu - \lambda + 1/2, 1 + 2\mu; z); \\ W_{\lambda,\mu}(z) &= e^{-z/2} z^{\mu+1/2} U(\mu - \lambda + 1/2, 1 + 2\mu; z). \end{aligned} \tag{37}$$

Is some special cases when $\kappa = \nu/2$ the Whittaker functions can formally be expanded with other functions (e.g. Bessel, Err) when $\{c_1 : c_1 \in \mathbf{N}\backslash\{-2, -4\}\}$. It is easy to show with the help of asymptotic forms that the velocity field $u \sim t^{-1/3}[M \mathrm{or} W(\cdot, \cdot; t^{-4/3})]$ decays for sufficiently large time which is a physical property of a viscous fluid. (It is worthwhile to mention, we found additional closed solutions only for $n = 1/2$ and for $n = 3/2$ from Eq. (33,35) for the density and velocity field which contain the HeunT functions, in a confusingly complicated expression.)

Now we compare our recent results to the former non-compressible ones. In the non-compressible case of the three dimensional NS equation, all the exponents have the $1/2$ value - like in the regular diffusion equation - except the decaying exponent of the pressure field which is 1. For non-compressible fluids the x component of the velocity field is described with the help of the Kummer functions

$$\tilde{g}(\eta) = c_1 U\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\eta + c)^2}{6\nu}\right) + c_2 M\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\eta + c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}, \tag{38}$$

where c_1 and c_2 are the usual integration constants. The viscosity is ν the external field is a and the c is the non-zero integration constant from the continuity equation - which can be set to zero. Additional properties of this formula was analyzed in our last study[7] in depth.

Figure 7 compares the regular parts of the solutions of Eq. (36) and Eq. (38) with the same viscosity value $\nu = 0.1$ and for $\tilde{c}_1 = 1, \tilde{c}_2 = 1, c_1 = 0$. The compressible parameters are $\kappa = 1$ and $n = 2$. Note that the shape function of velocity of the compressible flow has a maximum and a quick decay, the incompressible velocity shape function has no decay. However, these are the reduced one dimensional shape functions, and the total thee dimensional velocity fields have proper time decay for large time as it should be. The c_1 in the Whittaker function cannot be negative because it comes from the density equation. If it is zero or any other positive number plays no difference in the form of the shape function.

Figure 8 presents how the regular part of the solution Eq. (36) depends on the compressibility for a given value of viscosity. Note, the higher the compressibility the lower the maximum of the top speed of the system.

As a second case study Figure 9 presents how the regular part of the solution Eq. (36) depends on the viscosity for a given value of compressibility. Higher the viscosity the higher the maximal reached speed and the range of the system. In our investigation the role of the two viscosities cannot be separated from each other therefore this effect cannot be seen more clearly.

Note, that Eq. (36) is not a direct limit of Eq. (38) just a very similar one. More technical details of the calculations can be found in [24].

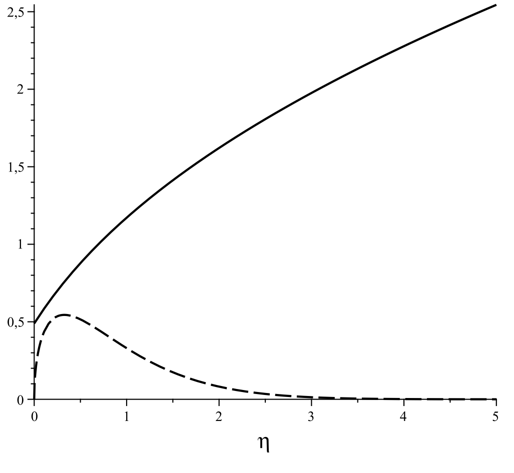

Figure 7. Comparison of the regular solutions for the non-compressible (solid line) Eq. (38) and the compressible case(dashed line) Eq. (36). The viscosities have the same numerical value $\mu = 0.1$.

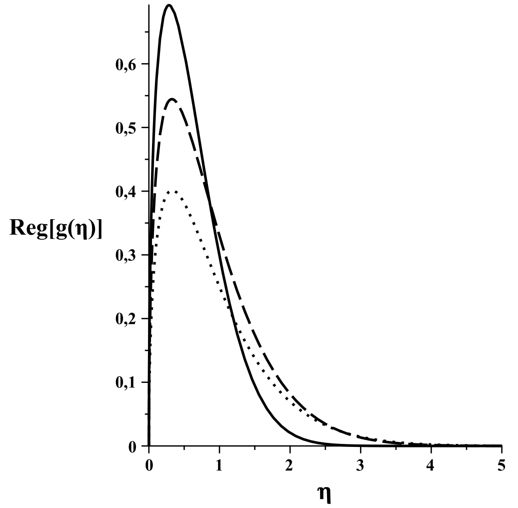

Figure 8. The compressibility dependence of the regular solution of Eq. (36) for $n = 2$ and for $\nu = 0.1$ viscosity. The solid line is for $\kappa = 0.1$ the dotted line is for $\kappa = 1$ and the dashed line is for $\kappa = 2$.

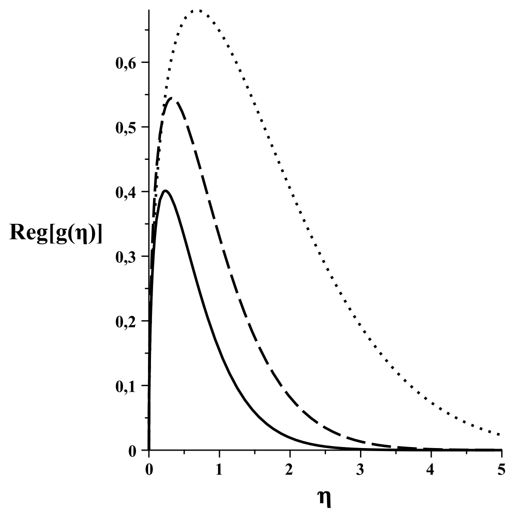

Figure 9. The viscosity dependence of the regular solution of Eq. (36) for $n = 2$ and $\kappa = 1$. The solid line is for $\nu = 0.05$ the dotted line is for $\nu = 0.1$ and the dashed line is for $\nu = 0.5$.

4. Incompressible Non-Newtonian Fluid

Dynamical analysis of viscous fluids is a never-ending crucial problem. A large part of real fluids do not strictly follow Newtons law and are aptly called non-Newtonian fluids. In most cases, fluids can be described by more complicated governing rules, which means that the viscosity has some additional density, velocity or temperature dependence or even all of them. General introductions to the physics of non-Newtonian fluid can be found in [25, 26]. In the following, we will examine the properties of a Ladyzenskaya type non-Newtonian fluid [27, 28]. Additional temperature or density dependent viscosities will not be considered in the recent study. There are some analytical studies available for non-Newtonian flows in connections with the boundary layer theory, which shows some similarity to our recent problem [29, 30]. The heat transfer in the boundary layer of a non-Newtonian Ostwald-de Waele power law fluid was investigated with self-similar Ansatz in details by [31, 32].

The Ladyzenskaya [27] model of non-Newtonian fluid dynamics can be formulated in the general vectorial form of

$$\begin{aligned}
\rho\frac{\partial \mathbf{u}_i}{\partial t} &+ \rho \mathbf{u}_j \frac{\partial \mathbf{u}_i}{\partial x_j} = -\frac{p}{\partial \mathbf{x}_i} + \frac{\partial \Gamma_{ij}}{\partial \mathbf{x}_j} + \rho \mathbf{a}_i \\
\frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_j} &= 0 \\
\Gamma_{ij} &\overset{Def}{=} (\mu_0 + \mu_1 |E(\nabla \mathbf{u})|^r) E_{ij}(\nabla \mathbf{u}) \\
E_{ij}(\nabla \mathbf{u}) &\overset{Def}{=} \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_i}\right)
\end{aligned} \tag{39}$$

where $\rho, \mathbf{u}_i, p, \mathbf{a}_i, \mu_0, \mu_1, r$ are the density, the two dimensional velocity field, the pressure, the external force, the dynamical viscosity, the consistency index and the flow behavior index. The last one is a dimensionless parameter of the flow. The E_{ij} is the Newtonian linear stress tensor, where **x**(x,y) are the Cartesian coordinates. The usual Einstein summation is used for the j subscript. In our next model, the exponent should be $r > -1$. This general description incorporates the next five different fluid models:

$$\begin{aligned}
\text{Newtonian for} \quad & \mu_0 > 0, \mu_1 = 0, \\
\text{Rabinowitsch for} \quad & \mu_0, \mu_1 > 0, r = 2, \\
\text{Ellis for} \quad & \mu_0, \mu_1 > 0, r > 0, \\
\text{Ostwald-de Waele for} \quad & \mu_0 = 0, \mu_1 > 0, r > -1, \\
\text{Bingham for} \quad & \mu_0, \mu_1 > 0, r = -1.
\end{aligned} \tag{40}$$

For $\mu_0 = 0$ if $r < 0$ then it is called a pseudo-plastic fluid, if $r > 0$ it is a dilatant fluid [26]. In pseudoplastic or shear thinning fluid the apparent viscosity decreases with increased stress. Examples are: blood, some silicone oils, some silicone coatings, paper pulp in water, nail polish, whipped cream, ketchup, molasses, syrups, latex paint, ice . For the paper pulp the numerical Ostwald-de Waele parameters are $\mu_1 = 0.418, r = -0.425$ [26]. In shear thickening or dilatant fluid, the apparent viscosity increases with increased stress. Typical examples are sand in water or suspensions of corn starch in water (sometimes called oobleck). There are numerous videos available on most popular video sharing portal where young people having fun with a pool full of oobleck .

The external force will be zero in our investigation $\mathbf{a_i} = 0$. In two dimensions, the absolute value of the stress tensor reads:

$$|E| = [u_x^2 + v_y^2 + 1/2(u_y + v_x)^2]^{1/2}, \tag{41}$$

where the $\mathbf{u}(u, v)$ coordinate notation is used from now on. (Note, that in three dimensions the absolute value of the stress tensor would be much complicated containing six terms instead of three.) Introducing the following compact notation

$$L = \mu_0 + \mu_1 |E|^r, \tag{42}$$

our complete two dimensional NS system can be defined much shorter:

$$\begin{aligned} u_x + v_y &= 0, \\ u_t + uu_x + vu_y &= -p_x/\rho + L_x u_x + L u_{xx} + \frac{L_y}{2}(u_y + v_x) + \frac{L}{2}(u_{yy} + v_{xy}), \\ v_t + uv_x + vv_y &= -p_y/\rho + L_y v_y + L v_{yy} + \frac{L_x}{2}(u_y + v_x) + \frac{L}{2}(v_{xx} + u_{xy}). \end{aligned} \tag{43}$$

Every two dimensional flow problem can be reformulated with the help of the stream function Ψ via $u = \Psi_y$ and $v = -\Psi_x$, which automatically fulfills the continuity equation. The system of (43) is now reduced to the following two PDEs

$$\begin{aligned} \Psi_{yt} + \Psi_y \Psi_{yx} - \Psi_x \Psi_{yy} &= -\frac{p_x}{\rho} + (L\Psi_{yx})_x + \left[\frac{L}{2}(\Psi_{yy} - \Psi_{xx})\right]_y \\ -\Psi_{xt} - \Psi_y \Psi_{xx} + \Psi_x \Psi_{xy} &= -\frac{p_y}{\rho} + \left[\frac{L}{2}(\Psi_{yy} - \Psi_{xx})\right]_x - (L\Psi_{yx})_y \end{aligned} \tag{44}$$

with $L = \mu_0 + \mu_1 \left[2\Psi_{xy}^2 + \frac{1}{2}(\Psi_{yy} - \Psi_{xx})^2\right]^{r/2}$.

Now, search the solution of this PDE system such as: $\Psi = t^{-\alpha} f(\eta),\ p = t^{-\epsilon} h(\eta),\ \eta = \frac{x+y}{t^\beta}$, where all the exponents α, β, γ are real numbers.

Unfortunately, the constraints, which should fix the values of the exponents become contradictory, therefore no unambiguous ODE can be formulated. This means that the new PDE system with the stream function does not have self-similar solutions. In other words the stream function has no diffusive property. This is a very instructive example of the applicability of the trial function of (1).

Let's return to the original system of (43) and use the trial function of

$$u = t^{-\alpha} f(\eta), \qquad v = t^{-\delta} g(\eta), \qquad p = t^{-\epsilon} h(\eta), \qquad \eta = \frac{x+y}{t^\beta}. \tag{45}$$

The next step is to determine the exponents. From the continuity equation we simple get arbitrary β and $\delta = \alpha$ relations. The two NS dictate additional constraints. (We skip the trivial case of $\mu_0 \neq 0, \mu_1 = 0$, which was examined in our former paper as the Newtonian fluid. [17]) Finally, we get

$$\mu_0 = 0,\ \ \mu_1 \neq 0,\ \ \alpha = \delta = (1 + r)/2,\ \ \beta = (1 - r)/2,\ \ \epsilon = r + 1. \tag{46}$$

Note, that r remains free, which describes various fluids with diverse physical properties, this meets our expectations. For the Newtonian NS equation, there is no such free parameter and in our former investigation we got fixed exponents with a value of $1/2$. For a physically relevant solutions, which is spreading and decaying in time all the exponents Eq. (46) have to be positive determining the $-1 < r < 1$ range. This can be understood as a kind of restricted Ostwald-de Waele-type fluid. After some algebraic manipulation a second order non-autonomous non-linear ODE remains

$$\mu_1(1+r)f''[2f']^{r/2} + \frac{(1-r)}{2}\eta f' + \frac{(1+r)}{2}f = 0, \tag{47}$$

where prime means derivation with respect to η. Note that for the numerical value $r = 0$ we get back the ODE of the Newtonian NS equation for two dimensions. In three dimensions the ODE reads:

$$9\mu_0 f'' - 3(\eta + c)f' + \frac{3}{2}f(\eta) - \frac{c}{2} + a = 0. \tag{48}$$

Its solutions are the Kummer functions [16]

$$f = c_1 \cdot KummerU\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\eta+c)^2}{6\mu_0}\right) + \\ c_2 \cdot KummerM\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\eta+c)^2}{6\mu_0}\right) + \frac{c}{3} - \frac{2a}{3}, \tag{49}$$

where c_1 and c_2 are integration constants, c is the mass flow rate, and a is the external field. These functions have no compact support. The corresponding velocity component however, decays for large time like $v \sim 1/t$ for $t \to \infty$, which makes these results physically reasonable. A detailed analysis of (49) was presented in [17].

Unfortunately, we found no integrating factor or analytic solution for (47) at arbitrary values of r. We mention that with using the symmetry properties of the Ansatz $u_{xx} = u_{xy} = u_{yy} = -v_{xx} = -v_{xy} = -v_{yy}$ the following closed form can be derived for the pressure field

$$h = \rho(\mu_1 2^{r/2+1} f'^{r+1} + f\eta - \tilde{c}_1 f) + \tilde{c}_2, \tag{50}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are the usual integration constants. The transition theorem states, that a second order ODE is always equivalent to a first order ODE system. Let us substitute $f' = l, f'' = l'$, then

$$\begin{aligned} f' &= l, \\ l' &= -\left(\frac{(1-r)}{2}\eta l + \frac{(1+r)}{2}f\right) / \left(\mu_1(1+r)2^{r/2}l^r\right), \end{aligned} \tag{51}$$

where prime still means derivation with respect to η. This ODE system is still non-autonomous and there is no general theory to investigate such phase portraits. We can divide the second equation of (51) by the first one to get a new ODE, where the former independent variable η becomes a free real parameter

$$\frac{dl}{df} = -\left(\frac{(1-r)}{2}\eta l + \frac{(1+r)}{2}f\right) / \left(\mu_1(1+r)2^{r/2}l^{r+1}\right). \tag{52}$$

Figure 10 shows the phase portrait diagram of (52) for water pulp the material parameters are $r = -0.425$ and $\mu_1 = 0.18$. We consider $\eta = 0.03$ as the "general time variable" to be positive as well. With the knowledge of the exponent range $-1 < r < +1$, two general properties of the phase space can be understood by analyzing Eq. (52):

Firstly, the derivative df/dl is zero at zero nominator values, which means $l = -(1 + r)/(1 - r)f/\eta$. This one is a straight line passing through the origin with gradient of $0 < (1+r)/(1-r) < \infty$ for $-1 < r < +1$. On Figure 10, the numerical value of the gradient is $-0.403/\eta = -13.5$.

Secondly, the derivative df/dl or the direction field is not defined for any negative l values because the power function $l^{-(r+1)}$ in the denominator is not defined for negative l arguments. It is not possible to extract a non-integer root from a negative number. The denominator is always positive.

These properties dictate that there are two kinds of possible trajectories or solutions exist in the phase space. One type has a compact support, and the other has a finite range. We may consider the x axis as the velocity $f \sim v(\eta)$ and y axis as the $f' \sim a(\eta)$ acceleration for a fixed scaled time $\eta = const$. It also means that the possible velocity and accelerations for a general time cannot be independent from each other. The factors of the second derivative f'' show some similarity to the porous media equation, where the diffusion coefficient has also an exponent. This is the essential original responsibility for the solution with compact support [34]. Additional numerical solutions of Eq. (47) obtained with the help of the Briot-Bouquet theorem was presented at the end of [33].

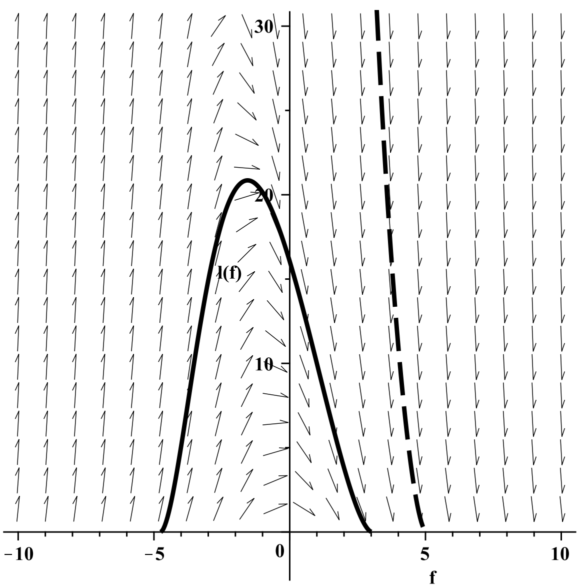

Figure 10. The phase portrait of Eq. (52) for $\eta = 0.03, r = -0.425$ and $\mu_1 = 0.18$. Two different kind of trajectories are presented: one with compact support (solid line), and the other one with compact range (dashed line).

Our main result is that the velocity field of the fluid - in contrast to our former Newtonian result - has a compact support, which is the major difference. We can explain it with the following everyday example: Let's consider two pots in the kitchen, one is filled with water and the other is filled with chocolate cream. Start to stir both with a wooden spoon in the middle, after a while the whole mass of the water becomes to move due to the Newtonian viscosity, however the chocolate cream far from the spoon remains stopped even after a long time.

5. Incompressible Newtonian Fluid with Heat Conduction

In the following we analyze the dynamics of a two-dimensional viscous fluid which is coupled to heat conduction. Such systems were first investigated by Boussinesq [35] and Oberbeck [36] in the nineteenth century. Oberbeck used a finite series expansion. He developed a model to study the heat convection in fluids taking into account the flow of the fluid as a result of temperature difference. He applied the model to the normal atmosphere. More detailes of the Boussinesq approximation can be found our in original paper [37].

More than half a century later Saltzman [38] tried to solve the same model with the help of Fourier series. At the same time Lorenz [39] analyzed the solutions with computers and published the plot of a strange attractor which was a pioneering results and the advent the studies of chaotic dynamical systems. The literature of chaotic dynamics became enormous, however a modern basic introduction can be found in [40].

Later, till to the first beginning years of the millennium [39] Lorenz analyzed the final first order chaotic ordinary differential equation(ODE) system with different numerical methods. This ODE system becomes an emblematic object of chaotic systems and attracts much interest till today [41].

On the other side critical studies came to light which go beyond the simplest truncated Fourier series. Curry for example gives a transparent proof that the finite dimensional approximations have bounded solutions [42]. Musielak *et al* [43] in three papers analyzed large number of truncated systems with different kinds and found chaotic and periodic solutions as well.

In the next we apply a completely different investigation approach, the self-similar Ansatz. We investigated one dimensional Euler equations with heat conduction as well [33] which can be understood as the precursor of the recent study. To our knowledge this kind of investigation method was not yet applied to the Oberbeck-Boussinesq (OB) system. Our major result is that the temperature field shows a strongly damped single periodic oscillation which can mimic the appearance of Rayleigh-Bénard convection cells. The question how our results connected to general chaotic and turbulence conception like intermittency, enstropy are discussed in our original study [37] as well.

The start with the original system of [38]

$$\begin{aligned} u_t + uu_x + wu_z + P_x - \nu\left(u_{xx} + u_{zz}\right) &= 0, \\ w_t + uw_x + ww_z + P_z - eGT_1 - \nu\left(w_{xx} + w_{zz}\right) &= 0, \\ T^1_t + uT^1_x + wT^1_z - \kappa(T^1_{xx} + T^1_{zz}) &= 0, \\ u_x + w_z &= 0, \end{aligned} \tag{53}$$

where $u, w,$ denote respectively the x and z velocity coordinates, T^1 is the temperature difference relative to the average ($T^1 = T - T_{av}$) and P is the scaled pressure over the density . The free physical parameters are ν, e, G, κ kinematic viscosity, coefficient of volume expansion, acceleration of gravitation and coefficient of thermal diffusivity. The first two equations are the Navier-Stokes equations, the third one is the heat conduction equation and the last one is the continuity equation all are for two spatial dimensions. The Boussinesq approximation means the way how the heat conduction is coupled to the second NS equation. Chandrasekhar [44] presented a wide-ranging discussion of the physics and mathematics of Rayleigh-Benard convection along with many historical references.

Every two dimensional flow problem can be reformulated with the help of the stream function Ψ via $u = \Psi_y$ and $v = -\Psi_x$ which automatically fulfills the continuity equation. The subscripts mean partial derivations. After introducing dimensionless quantities the system of (53) is reduced to the next two PDEs

$$
\begin{aligned}
(\Psi_{xx} + \Psi_{zz})_t + \Psi_x(\Psi_{xxz} + \Psi_{zzz}) - \Psi_z(\Psi_{xxx} + \Psi_{zzx}) - \\
\sigma(\theta_x - \Psi_{xxxx} - \Psi_{zzzz} - 2\Psi_{xxzz}) = 0, \\
\theta_t + \Psi_x\theta_z - \Psi_z\theta_x - R\Psi_x - (\theta_{xx} + \theta_{zz}) = 0,
\end{aligned}
\tag{54}
$$

where Θ is the scaled temperature, $\sigma = \nu/\kappa$ is the Prandtl Number and $R = \frac{GeH^3\Delta T_0}{\kappa\nu}$ is the Rayleigh number and H is the height of the fluid. A detailed derivation of (54) can be found in [38].

All the mentioned studies are investigated these two PDEs with the help of some truncated Fourier series, different kind of truncations are available which result different ODE systems. The derivation of the final non-linear ODE system from the PDE system can be found in the original papers [38, 39]. Bergé et al. [45] contains a slightly different derivation of the Lorenz model equations, and in addition, provides more details on how the dynamics evolve as the reduced Rayleigh number changes. A detailed treatment of the Lorenz model can be found in the book of Sparrow [46]. Hilborn [47] presents the idea of the whole derivation in a transparent and clear way. Therefore, we skip this derivation.

Some truncations even violates energy conservation [41] and some not. Roy and Musiliak [43] in their exhausting three papers present various energy-conserving truncations. Some of them contain higher horizontal modes, some of them contain higher vertical modes and some of them both kind of modes in the truncations. All these models show different features some of them are chaotic and some of them - in well-defined parameter regimes - show periodic orbits in the projections of the phase space. This is somehow a true indication of the complex nature of the original flow problem. It is also clear that the Fourier expansion method which is a two hundred year old routine tool for linear PDEs fails for a relevant non-linear PDE system.

We may investigate both dynamical systems, the original hydrodynamical (53) or the other one (54) which is valid for the stream functions.

Similar to the former studies [38, 39] try to solve the PDEs for the dimensionless stream and temperature functions in the form of

$$
\Psi = t^{-\alpha} f(\eta), \; \theta = t^{-\epsilon} h(\eta), \; \eta = \frac{x+z}{t^\beta}. \tag{55}
$$

Unfortunately, after some algebra it becomes clear that the constraints which should fix the values of the exponents become contradictory, therefore no unambiguous ODE can be derived. This means that the PDE of the stream function and the dimensionless temperature do not have self-similar solutions. In other words these functions have no such a diffusive property which could be investigated with the self-similar Ansats, which is a very instructive example of the applicability of the trial function of (55). Our experience shows that, most of the investigated PDEs have a self-similar ODE system and this is a remarkable exception.

Now investigate the original hydrodynamical system with the next Ansatz

$$u(\eta) = t^{-\alpha} f(\eta), \;\; w(\eta) = t^{-\delta} g(\eta), \;\; P(\eta) = t^{-\epsilon} h(\eta), \;\; T^1(\eta) = t^{-\omega} l(\eta), \tag{56}$$

where the new variable is the usual $\eta = (x+z)/t^{\beta}$. All the five exponents $\alpha, \beta, \delta, \epsilon, \omega$ are

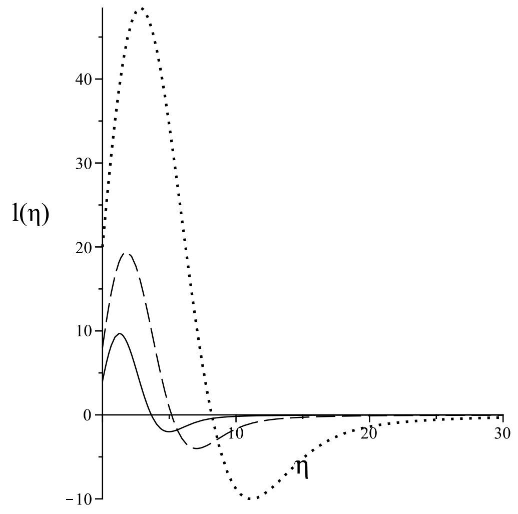

Figure 11. Different shape functions of the temperature Eq. (59) as a function of η for different thermal diffusivity. The integration constants are $c_1 = c_2 = 1$ the same for all the three curves. The solid the dashed and the dotted lines are for $\kappa = 1, 2, 5$, respectively.

real numbers. The f, g, h, l objects are the shape functions of the corresponding dynamical variables.

After some algebraic manipulations the following constrains are fixed among the self-similarity exponents : $\alpha = \delta = \beta = 1/2$, $\epsilon = 1$ and $\omega = 3/2$ which are called the universality relations. Note, that all exponents have a fixed numerical value which simplifies the structure of the solutions. There is no free exponential parameter in the original dynamical system, like an exponent in a EOS for the compressible NS system.

These universality relations dictate the corresponding coupled ODE system which is the following

$$\begin{aligned} -\frac{f}{2} - \frac{f'\eta}{2} + ff' + gf' + h' - 2\nu f'' &= 0, \\ -\frac{g}{2} - \frac{g'\eta}{2} + fg' + gg' + h' - eGl - 2\nu g'' &= 0, \\ -\frac{3l}{2} - \frac{l'\eta}{2} + fl' + gl' - 2\kappa l'' &= 0, \\ f' + g' &= 0. \end{aligned} \tag{57}$$

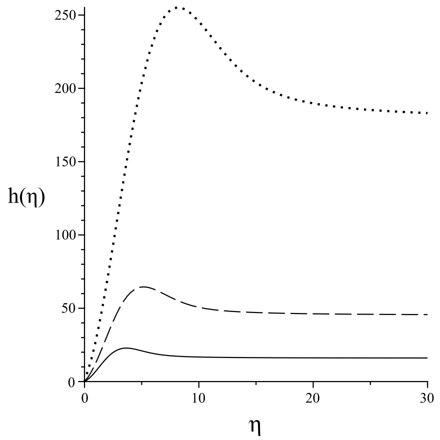

Figure 12. Different shape functions of the pressure Eq. (62) as a function of η for different thermal diffusivity. The integration constants are taken $c_1 = c_2 = 1$ for all the three curves. We fixed the value of $eG = 1$ as well. The solid, the dashed and the dotted lines are for $\kappa = 1, 2, 5$ numerical values, respectively.

From continuity equation we automatically get the $f+g = c$ and $f''+g'' = 0$ conditions which are necessary for the solutions. For the shake of simplicity we consider the $c = 0$ case in the following. If $c \neq 0$ then the Eq. (58) is slightly modified and the results are the KummerM and KummerU functions, but the shape of the functions remains the same which is crucial for the forthcoming analysis. After some algebraic manipulation the next single ODE for the shape function of the temperature distribution can be separated

$$2\kappa l'' + \frac{l'\eta}{2} + \frac{3l}{2} = 0. \tag{58}$$

After a quadrature the solution is

$$l = c_1 \left[4erfi\left(\frac{\sqrt{2}\eta}{4\sqrt{\kappa}}\right)\sqrt{2\pi}\left(\kappa - \frac{\eta^2}{4}\right)e^{-\frac{\eta^2}{8\kappa}} + 4\sqrt{\kappa}\eta \right] + c_2 e^{-\frac{\eta^2}{8\kappa}}(4\kappa - \eta^2) \tag{59}$$

where c_1, c_2 are the usual free integration constants. The erfi means the imaginary error function defined via the integral $2/\sqrt{\pi}\int_0^x exp(x^2)dx$ for more details see [16]. It is interesting, that the temperature distribution is separated from the other three dynamical variables an does not depend on the viscosity coefficients as well. We may say, that among the solution obtained from the self-similar Ansazt the temperature has the highest priority and this quantity defines the pressure and the velocity field. That is a remarkable feature. In a former study, where the one-dimensional Euler system was investigated with heat conduction [48] we found the opposite property, the density and the velocity field were much simpler than the temperature field. Figure 11 presents different shape functions of the temperature for different thermal diffusivity values. The first message is clear, the larger the thermal diffusivity the larger the shape function of the temperature distribution. A detailed analysis of Eq. (59) shows that for any reasonable κ and c values the main property of the function is not changing - has one global maximum and minimum with a strong decay for large ηs. A second remarkable feature is the single oscillation which is not a typical behavior for self-similar solutions. We investigated numerous non-linear PDE systems till today [17, 24, 33, 48] some of them are even not hydrodynamical [6, 49] and never found such a property. This analysis clearly shows that at least the temperature distribution in this physical system has a single-period anharmonic oscillation.

For a fixed time value "t" and a well-chosen coordinate "z" the difference of values of $\eta = (x+z)/t^{\beta}$ yields a minima and a maxima corresponds to that Δx at which the temperature (and density) fluctuation may start the Rayleig-Bénard convection. We will see later that with the same consideration (when time and z coordinate are fixed) the velocity components u and w are different at these x coordinate points therefore the inhomogenity is present which start the rotation of the Rayleig-Bénard cell. This is the main result of the recent study.

To go a step further we may calculate the Fourier transform of the shape function, Eq. (59) $l(\eta)$ it we consider η as a generalized time dependence we may get the generalized spectral distribution. (An analytic expression for the Fourier transform is available, which we skip now.) The first term (which is proportion to c_1) becomes a complex function, however the general overall shape remains the same, a single-period anharmonic oscillation with a global minimum and maximum like on Figure 11. Of course, the zero transition of the function depends on the value of κ. The second term of the Fourier transformed function which is proportional to c_2 remains a Gaussian which is not interesting.

For completeness we give the full two dimensional temperature field as follows

$$\begin{aligned} T_1(x,z,t) = \quad & c_1 \quad t^{-3/2}\left[4erfi\left(\frac{x+z}{4(\kappa t)^{1/2}}\right)\sqrt{2\pi}\left(\kappa - \frac{(x+z)^2}{4t}\right)e^{-\frac{(x+z)^2}{8\kappa t}} + \frac{4\sqrt{\kappa}(x+z)}{t^{1/2}}\right] + \\ & c_2 \quad t^{-3/2}e^{-\frac{(x+z)^2}{8\kappa t}}\left(4\kappa - \frac{(x+z)^2}{t}\right). \end{aligned} \tag{60}$$

The shape function of the pressure field can be obtained from the temperature shape function via the following equation

$$h' = \frac{eGl}{2} \tag{61}$$

with a similar solution to (59)

$$h = c_1\left[2\kappa\sqrt{2\pi}eG\cdot erfi\left(\frac{\sqrt{2}\eta}{4\sqrt{\kappa}}\right)\eta e^{-\frac{\eta^2}{8\kappa}}\right] + c_2 2eG\kappa\eta e^{-\frac{\eta^2}{8\kappa}} + c_3, \tag{62}$$

this can be understood that the derivative of the pressure is proportional to the temperature (Figure 12). With the known numerical value of the exponent $\epsilon = 1$ the scaled pressure field can be expressed as well $P(x,z,t) = t^{-1}h([x+z]/t^{-1/2})$. Note, the difference between the ω and the ϵ exponents, which are responsible for the different asymptotic decays. The temperature field has a stronger damping for large η than the pressure field. (It is worth to mention that for the three dimensional NS equation, without any heat exchange the decay exponent of the pressure term is also different to the velocity field [17].)

At last the ODE for the shape function of the velocity component z reads

$$4\nu g'' + g'\eta + g + eGl = 0 \tag{63}$$

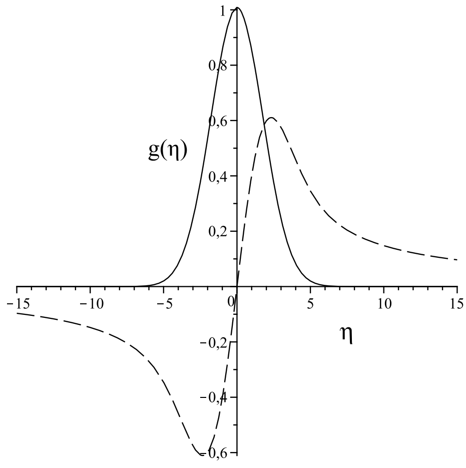

Figure 13. The shape functions of the z velocity field $g(\eta)$ Eq. (65) as a function of η. The solid line is the real and the dotted is the complex part. All the integration constants are taken $\tilde{c}_1 = \tilde{c}_2 = c_2 = 1$. The physical constant $eG = 1$ as well. The $\kappa = 0.04$ and $\nu = 0.8$.

which directly depends on the temperature on $l(\eta)$ and all the physical parameters ν, e, G, κ, of course. In contrast to the pressure and temperature field there is no closed solutions available for a general parameter set. The formal, most general solution is

$$g = \tilde{c}_2 e^{-\frac{\eta^2}{8\nu}} + e^{-\frac{\eta^2}{8\nu}} \left\{ \int \frac{1}{4\nu} \left[\left(\tilde{c}_1 - 4eG\kappa c_2 \eta e^{-\frac{\eta^2}{8\kappa}} - 4eG\kappa\sqrt{2\pi} c_1 erfi\left(\frac{\sqrt{2}\eta}{4\sqrt{\kappa}}\right) \eta e^{-\frac{\eta^2}{8\kappa}} \right) e^{\frac{\eta^2}{8\nu}} \right] d\eta \right\} \tag{64}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are the recent integration constants. Note, that the integral can be analytically evaluated if and only if $\nu = \kappa$ which is a great restriction to the physical system. We skip this solution now. The other way is to fix $c_1 = 0$ and let κ and μ free. The solution has the next form of

$$g = \tilde{c}_1 e^{-\frac{\eta^2}{8\nu}} erf\left(\frac{\eta}{4}\sqrt{-\frac{2}{\nu}}\right) + \tilde{c}_2 e^{-\frac{\eta^2}{8\nu}} - \frac{4eGc_2\kappa^2 e^{-\frac{\eta^2}{8\kappa}}}{\kappa - \nu}. \tag{65}$$

Note, that now the $\nu \neq \kappa$ condition is obtained. The $\tilde{c}_1$ and $\tilde{c}_2$ are the recent integration constants as above, it is interesting that if both of them are set to zero, the solution is still not trivial. For a physical system the kinematic viscosity $\nu > 0$ is always positive, therefore in the case of $\tilde{c}_1 \neq 0$ the solution becomes complex. Figure 14 shows the shape function of the z velocity component. It is clear that the real part is a Gaussian function and the complex part is a Gaussian distorted with an error function, which is an interesting final result. In the literature we can find system which shows similarities like the work of Ernst [50] who presented a study where a the asymptotic normalized velocity autocorrelation function calculated from the linearized Navier-Stokes equation has an error function shape.

To give coupling points to colleagues from other field, (like chaotic dynamics or turbulence) in the last section of the original publication [37] we calculated the enstropy of the system which is crucial quantity for two dimensional turbulence. The Fourier spectra of our velocity field was compared to other turbulence results as well. The question of fractional derivatives or fractal dimensions are addressed as well.

Further analysis is in progress. Recently, we try to generalize the original OB system (53) with temperature dependent viscosity $\nu(T)$ and heat conduction coefficients $\kappa(T)$. It seems that it is not possible and only the constant coefficients are available for self-similar analysis. To go beyond the Boussinesq approximation is also under consideration, the analysis is very exciting.

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