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Imre Ferenc Barna, Gabriella Bognár, László Mátyás, Mohamed Guedda, and Krisztián Hriczó



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# Analytic Solutions of the Two-dimensional Kardar-Parisi-Zhang Growing Equation

Imre Ferenc Barna<sup>1</sup>, Gabriella Bognár<sup>2,a)</sup>, László Mátyás<sup>3</sup>, Mohamed Guedda<sup>4</sup> and Krisztián Hriczó<sup>2</sup>

<sup>1</sup>Wigner Research Centre for Physics

Konkoly-Thege Miklós út 29 - 33, H-1121 Budapest, Hungary

<sup>2</sup>University of Miskolc, Miskolc-Egyetemváros 3515, Hungary

<sup>3</sup>Department of Bioengineering, Faculty of Economics, Socio-Human Sciences and Engineering, Sapientia Hungarian University of Transylvania Libertății sq. 1, 530104 Miercurea Ciuc, Romania

<sup>4</sup>Université de Picardie Jules Verne Amiens, Faculte de Mathematiques et d'Informatique, 33, rue Saint-Leu 80039 Amiens, France

a)v.bognar.gabriella@uni-miskolc.hu

**Abstract.** The two-dimensional Kardar-Parisi-Zhang dynamic interface growth equation is analyzed in Cartesian coordinates with different kind of trial-functions. We show that the one-dimensional self-similar Ansatz can be generalized for multi space dimensions in numerous ways leading to fine differences between the obtained results. The role of the noise term is discussed as well.

## INTRODUCTION

The basic physics of crystal growth and the dynamics of solidification fronts can be found in numerous textbooks [1]. It is also clear why this scientific field attracts much interest in the last decades. The simplest model which is capable to describe surface growth phenomena is the nonlinear generalization of the ubiquitous diffusion equation is the so called Kardar-Parisi-Zhang(KPZ) model

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u + \frac{\lambda}{2} (\nabla u)^2 + \eta(\mathbf{x}, t), \quad (1)$$

where  $u$  stands for the profile of the local growth [2],  $\nu$  and  $\lambda$  are constants. The first term on the right hand side describes relaxation of the interface by a surface tension preferring a smooth surface. The second term is the lowest-order nonlinear term that can appear in the surface growth equation justified with the Eden model. The last term on the left-hand side is a Langevin noise which mimics the stochastic nature of any growth process and has a Gaussian distribution usually. The scientific literature which deals with the KPZ equation became enormous in the last two decades. There are numerous studies available which try to enlighten both mathematical and physical properties of the equation. We refer to the some recent publications: [3, 4, 5].

## THEORY

In the following we investigate the two-dimensional KPZ equation of the form of

$$\frac{\partial u(x, y, t)}{\partial t} = \nu \left( \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \right) + \frac{\lambda}{2} \left( \frac{\partial u(x, y, t)}{\partial x} + \frac{\partial u(x, y, t)}{\partial y} \right)^2 + \eta(x, y, t). \quad (2)$$

It has been known for a long time that the one-dimensional self-similar Ansatz  $u(x, t) = t^{-\alpha} f(x/t^\beta)$  can be generalized for two spatial dimensions in many different ways [6]. Unfortunately, only the linear (we say trivial) generalization of

the Ansatz can be applied for fluid dynamical equations (like Euler or Navier-Stokes equations), where the material (or substantial or total) derivative of the fields are present [7].

The most general case has the form of

$$u(x, y, t) = t^{-\alpha} f\left(\frac{F(x, y)}{t^\beta}\right) \quad (3)$$

which is an implicit generalization of a two-dimensional curve if  $F(x, y) = 0$ . Hence an implicit curve can be considered as the set of zeros of a function of two variables. If  $F(x, y)$  is a polynomial in two variables, the corresponding curve is called an algebraic curve, and specific methods are available for studying it. In principle, in our case,  $F(x, y)$  can be any such curve with existing continuous first and second spatial derivatives dictated by the KPZ equation. This condition narrows the set of available functions, however, the remaining class of functions is still quite large. For the sake of completeness we mention that in three spatial dimensions  $F(x, y, z)$  could be understood as an implicit parametrization of a two-dimensional surface.

## THE SUMMATION OF THE RESULTS

We found no direct method to find and fix all the direct forms of the available  $F(x, y)$  functions. We want to know if a specific  $F(x, y)$  function is valid, then the time and spatial derivatives are evaluated and substituted in Eq. (2) resulting additional algebraic equations among the exponents  $\alpha$  and  $\beta$ . If these equations can be solved then a nonlinear second-order ordinary differential equation can be obtained which defines the solution. In some cases even closed form solutions can be evaluated. But if the algebraic equation leads to contradiction(s) then no ordinary differential equation (ODE) can be defined and the trial function cannot reduce the initial partial differential equation to an ODE.

In our study, we investigate the remaining three self-similar Ansätze in the two-dimensional KPZ equation with and without any additional noise term

$$u(x, y, t) = t^{-\alpha} f\left(\frac{a \cdot x + b \cdot y + c}{t^\beta}\right) = t^{-\alpha} f(\omega), \quad (4)$$

$$u(x, y, t) = t^{-\alpha} g\left(\frac{\sqrt{x^2 + y^2}}{t^\beta}\right), \quad (5)$$

$$u(x, y, t) = t^{-\alpha} h\left(\frac{[x - y]^2}{t^\beta}\right), \quad (6)$$

where  $\omega$  represents the all-time argument of the shape functions  $f$ ,  $g$  and  $h$ . The ODEs obtained using these three trial functions are of the following forms

$$f'' \nu(a^2 + b^2) + f' \left(\frac{\omega}{2} + \frac{\lambda}{2} f' [a^2 + b^2]\right) = 0, \quad (7)$$

$$f'' \nu \omega + f' \left(\frac{\omega^2}{2} + \frac{\lambda \omega f'}{2}\right) = 0, \quad (8)$$

$$8f'' \nu \omega + f' (\omega + 2\lambda \omega f' + 4\nu) = 0. \quad (9)$$

Note, there is a non-trivial difference between the first two ODEs. It is not trivial to find the most general case in respect to the exponents of  $x$  and  $y$ . We still look for additional solutions whose existence cannot be excluded.

Figure 1 and Fig. 2 compare the solutions of the two neighboring ODEs without any noise term. It is clear that without any kind of additional noise term  $\eta(\omega[x, y, t])$  the surface growing has a saturation and no extra structure is present. We may reveal that the major differences occur on the short time scale just after the start of the growing process.

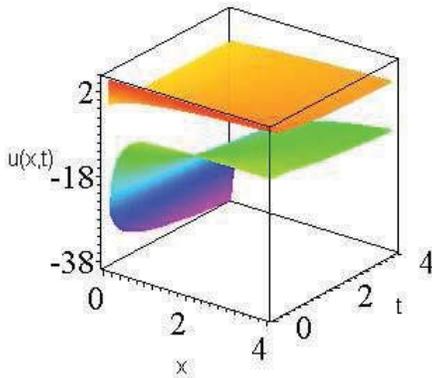
Figure 1 compares the solutions of Eq. (7) and Eq. (8). Both  $\lambda$  and  $\nu$  values were set to unity. For a better comparability, the  $a = b = 10$  values were taken. The upper brown surface represents the solution of Eq. (7) and the low-lying more spreading surface is for Eq. (8).

Figure 2 compares the results for Eq. (8) and (9). Both  $\lambda$  and  $\nu$  values were set to unity. For a better comparability the  $a = b = 10$  values were taken. The upper brown surface represents the solution of Eq. (8) and the low-lying more spreading surface is for Eq. (9).

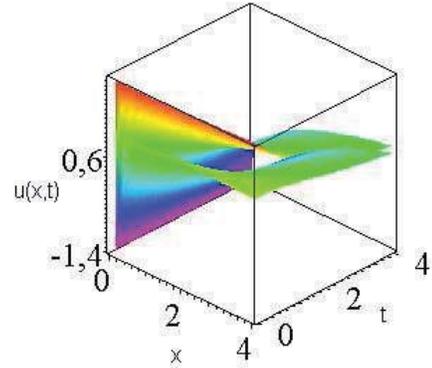
In our in-depth analysis, the effects of additional seven different kind of noise terms were investigated. Four of them are power-law type of noises:

- the brown noise  $\eta = a/\omega^2$ ,
- the pink noise  $\eta = a/\omega$ ,
- the white noise  $\eta = \text{constant}$ ,
- the  $\eta = a\omega$  blue noise.

The obtained results can be expressed with some special functions mostly with the Kummer M, U and the Whittaker M and W functions. Additionally, three other kind of noise functions were investigated, the Lorentzian and Gaussian and the periodic  $\cos(\omega)$  noise, as well. Unfortunately, there are no general formula available for all noise terms with arbitrary physical parameters  $\nu$  and  $\lambda$ . However, if it is possible to investigate all the physical properties of the results and to discuss the role of the free physical parameters.



**FIGURE 1.** The  $y = 0$  projection of the solutions of the KPZ equations Eq. (7) and Eq. (8)



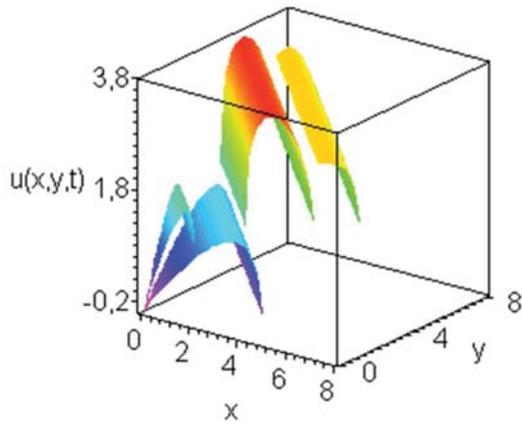
**FIGURE 2.** The  $y = 0$  projection of the solutions of the KPZ equations Eq. (8) and (9)

Figures 1 and 2 look much more similar. All parameters used were set the same in the numerical simulations. Figure 3 represents the solutions applying the first Ansatz (Eq. 4) and constant (white) noise, when the parameters are fixed at different times ( $t = 0.1, t = 5$ ). Figure 4 shows the solutions of the KPZ equation applying the first Ansatz (Eq. 4) and linear (blue) noise for fixed parameters and at different times ( $t = 0.1, t = 1$ ).

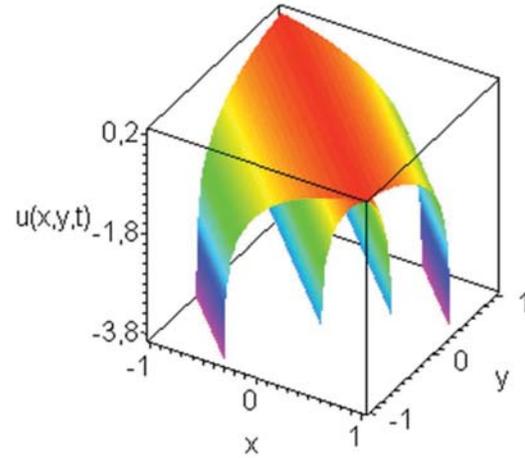
## Summary

Numerical simulations have been carried out by using MAPLE to obtain solutions to KPZ equation with different noise terms. On the base of our analytical investigations, the following results can be formulated:

- the possible one-dimensional self-similar Ansatz can be generalized for two-dimensional cases,
- we have found three different kind of Ansatz applicable for numerical and analytical investigations of the two-dimensional KPZ equation,
- various noise terms to Eq. (3) have been analyzed, and we found that the number of analytic solutions are much less in two-dimensions than in one spatial dimension.



**FIGURE 3.** The solutions of the KPZ equation (Eq. 4) with constant (white) noise at  $t = 0.1$  and  $t = 5$



**FIGURE 4.** The solutions of the KPZ equation applying the first Ansatz (Eq. 4) with linear (blue) noise at  $t = 0.1$ ,  $t = 1$

### ACKNOWLEDGMENTS

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